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Emerging social brain: A collective self-motivated Boltzmann machine

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ABSTRACT

Boltzmann machines are unsupervised-learning neural networks, which have contributed to the opening of the field of deep learning architectures. Here we show that, using the modern theory of economic growth, when the number of agents in a free-market society with equal opportunity exceeds a threshold value, a Boltzmann-like income distribution emerges, where the entropy plays the role of swarm intelligence in humans and quantifies its cumulative technological progress. Theoretically, we further show that the emergence of a Boltzmann-like income distribution in a society of optimizing agents reflects the spontaneous organization of a human society to form a Boltzmann machine in which each person plays a role analogous to that of a neuron within a brain-like architecture. This Boltzmann machine exhibits three essential brain-like features, namely the McCulloch-Pitts learning rule, unsupervised-learning, and self-motivation, and satisfies in addition the minimum free-energy principle of the brain theory. Empirically, we investigate the household income data from 66 free-market countries and Hong Kong SAR, and find that, for all of the countries, the income structure for low and middle classes (about 95% of populations) is accurately described by a Boltzmann-like distribution. We suggest that this is a statistical signature that our social networks are going through a critical evolution in the form of a kind of brain-like structure.

1. Introduction

The emergence of intelligence is a mysterious phenomenon. In the past, intelligence was often considered as a special ability of higher organisms. However, the swarm intelligence in slime molds put into question such reductionism [1]. Although slime molds are single-celled brainless organisms [1,2], through interactions depending on external conditions, they have the ability to find the minimum-length solution between two points in a labyrinth. The phenomena of swarm intelligence have been found in many biological colonies [3,4], such as ants [5–7], bees [6, 8, 9], birds [10,11], fishes [12–14], humans [4, 15–18] and so on. Inspired by the notion of swarm intelligence, some scholars even argue that "collective minds" might emerge from biological colonies [19,20]. In this vein, the function of the human brain seems to be a swarm intelligence of neurons: to the best of our knowledge, none of the neurons can intelligently think something, but the brain, which consists of tens of billions of neurons, can. From this perspective, the phenomenon of swarm intelligence sheds new light on understanding the operation of the brain; therefore, it is meaningful to investigate how swarm intelligence emerges from biological colonies.

To explore swarm intelligence, let us first consider ants' behaviors (see Fig. 1a), which are quite simple. They communicate with each other by using tentacles and via pheromones (see Fig. 1b), and each one has arguably meager intelligence. However, as the number of ants increases, they can do some (what can be deemed to be) intelligent activities (see Fig. 1c). Specifically, as long as the number of ants exceeds a threshold value, they will display highly complex and apparently intelligent behavior, e.g., building a bridge in the air (see Fig. 1d). By observing the behaviors of ants, we suggest that the swarm intelligence is a phenomenon of quantitative accumulation leading to qualitative transformation. We hypothesize that, when the number of individuals in a biological colony exceeds a sufficiently large threshold value, the colony will exhibit global intelligent behaviors. In a similar way, Fig. 1e shows that, to

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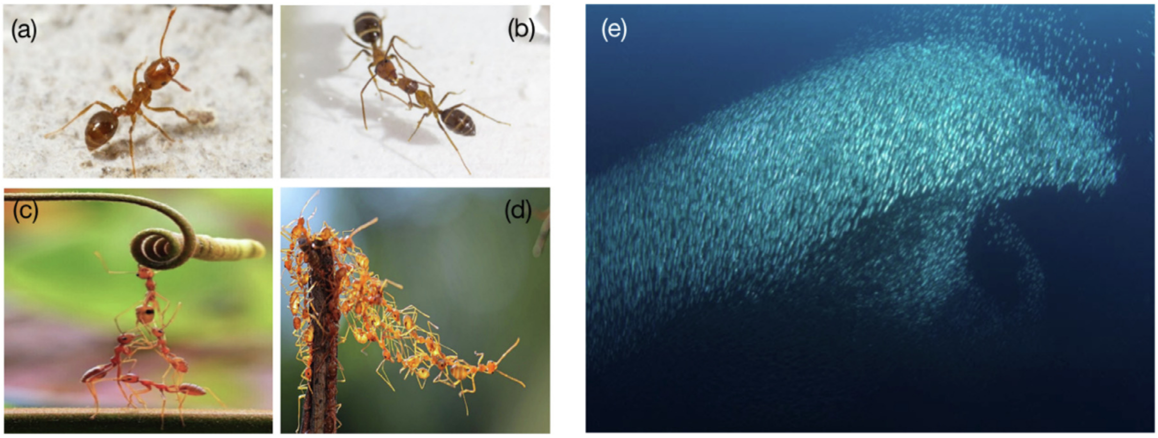

Fig. 1. Quantitative accumulation leads to swarm intelligence. **a:** The behaviors of ants are quite simple, and one ant has limited intelligence. **b**: Ants communicate with one another by using tentacles and by releasing pheromones. **c**: As the number of ants increases, they can perform intelligent activities. **d**: As long as the number of ants exceeds a sufficiently large threshold value, they will display highly complex and apparently intelligent behaviors, e.g., building a bridge in the air. **e:** Quantitative accumulation leading to swarm intelligence is a universal phenomenon. To confuse the potential predators, thousands of sardines "intelligently" array themselves to disguise as a "dolphin".

confuse the potential predators (such as sharks), thousands of sardines "intelligently" array themselves to disguise as a "dolphin". As far as we know, each sardine probably does not understand why they have constituted a "dolphin".

Inspired by the four examples of slime molds, neurons, ants and sardines, we now propose a mathematical model of the mechanism by which quantitative accumulation can lead to swarm intelligence. In this paper, we apply this formalism to humans and the emergence of collective intelligence in human societies.

2. Two-parameter Boltzmann-like distribution

Like other biotic populations, humans, who are lying at the top of the biological chain, also exhibit swarm intelligence as long as the number of individuals exceeds a sufficiently large threshold value. To see this, we use the modern theory of economic growth, where gross domestic product (GDP) of a human society can be written in the form [21]:

$$\mathrm{GDP} = E(N(L, K), T), \tag{1}$$

where $N(L, K)$ denotes the number of agents, L denotes the amount of labor, K denotes the stock of capital, and T denotes the technological factor of the society. The productions and exchanges among $N(L, K)$ agents can be simulated by Arrow-Debreu general equilibrium model [22], which is a system of procedural justice [23,24]. This model describes well a competitive economy with equal opportunity, just as a Blockchain economy. Based on this model, Tao [24–26] showed that, as an evolutionary result of natural selection, when the number of agents, $N(L, K)$, exceeds a threshold value N_0, a Boltzmann-like income distribution will emerge:

$$\left\{ \begin{array}{l} a_i = e^{-\frac{\varepsilon_i - \mu}{\theta}} \\ \varepsilon_i \geq \mu \\ i = 1, 2, \ldots, n \end{array} \right. , \tag{2}$$

where a_i denotes that there are a_i agents, each of which obtains ε_i units of income, and $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$. Here, the threshold value N_0 is a sufficiently large number determined in [24–26]. Furthermore, one has by definition $N(L, K) = \sum_{i=1}^{n} a_i$ and $E(N(L, K), T) = \sum_{i=1}^{n} a_i \varepsilon_i$. The parameters $\mu = \partial E / \partial N$ and $\theta = \partial E / \partial T$ obey a self-referential equation [18]:

$$N \frac{\partial E(N, T)}{\partial N} + (T - N) \frac{\partial E(N, T)}{\partial T} = E(N, T), \tag{3}$$

which guarantees that the technological factor T is self-referential [27–29], see Supporting Information (SI). By solving Eq. (3), T obeys the following form [18]:

$$T = \ln \Omega\left(\{a_i\}_{i=1}^{n}\right) - \ln N!. \tag{4}$$

Here $\Omega(\{a_i\}_{i=1}^{n})$ denotes the size of the set of N agents' permissible collective strategies [24], i.e., these N agents can freely and collectively take any option among $\Omega(\{a_i\}_{i=1}^{n})$ consensus strategies, where $N \geq N_0$. The mathematical form of Eq. (4) shows that the technological factor T can be interpreted as the entropy of a human society as well [24,25]. The factor $\ln N!$ in Eq. (4) is the famous Gibbs term. This indicates that the Boltzmann-like distribution (2) is similar to a quantum distribution, which differs from the Boltzmann distribution as described in a classical context by Dragulescu and Yakovenko [30]. The presence of the Gibbs term is due to the self-reference nature of the entropy in a human society. Later, we will show that the Gibbs term drives a human society to become a self-motivated system.

3. Swarm intelligence and entropy

We propose to interpret technological progress in a society as equivalent to improving swarm intelligence in humans. Hence, by Eq. (4), the so-called emerging intelligence comes simply from many more options of collective strategies as long as the number of individuals, N, exceeds the threshold value N_0. From this standpoint, swarm intelligence actually aims to maximize the freedom of options in collective decisions [24,31]. This is also in agreement with the concept that self-organization in complex systems [32–42] can be treated as decision making (as it is performed by humans) and, vice versa, decision making is nothing but a kind of self-organization in the decision maker nervous systems [17]. Indeed, self-organization can be mapped onto the process of evaluating the probabilities of macroscopic states or, equivalently, of prospects in the search for a state with the largest probability. And, the general way of deriving the probability measure for classical systems is the principle of minimal information, that is, the conditional entropy maximization under given constraints. Moreover, related combinatorial arguments have been used to explain the J-shape of the acceleration innovations [43]. Furthermore, swarm intelligence in humans is supported by the emerging market intelligence hypothesis [16], which stresses that the collective intelligence dwarfs the individual ones.

On the other hand, some scientists proposed that human cooperation can be regarded as a manifestation of swarm intelligence in humans, and in this vein there is a literature relating human cooperation to phase transition in the sense of statistical physics [44,45]. In the next section, we will show that, if the income structure of a society obeys the Boltzmann-like income distribution (2), this society will form a Boltzmann machine, which can be regarded as a special version of the Ising model. It is well-known that the latter may have phase transition. This leads to the potential of studying phase transitions occurring in a human society.

4. Spontaneous emergence of a Boltzmann machine

Since the Boltzmann-like distribution (2) is associated with the emergence of technology (4), itself related to swarm intelligence, we investigate how it affects the behaviors of humans. We first show that the Boltzmann-like income distribution (2) will spontaneously induce a human society to form a Boltzmann machine [46–51]. To derive the Boltzmann machine, we assume that the GDP, E, should be a function of the state variables of agents. Let us denote the state variables of N agents by the vector $(\boldsymbol{h},\boldsymbol{v})=(h_1,\dots,h_n,v_1,\dots,v_m)$, where $n+m=N$. For example, we can use $h_i=1$ (or $v_j=1$) to denote that the agent i (or agent j) is active in markets, and $h_i=0$ (or $v_j=0$) to denote inactivity. Thus, without loss of generality, the GDP $E=E(\boldsymbol{h},\boldsymbol{v})$ can be expanded as the Taylor's series:

$$E(\boldsymbol{h},\boldsymbol{v})=E(0,0)+E_1(\boldsymbol{h},\boldsymbol{v}), \tag{5}$$

with

$$E_1(\boldsymbol{h},\boldsymbol{v})=\sum_{i=1}^{n}\sum_{j=1}^{m}\omega_{ij}h_iv_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}h_ih_j+\sum_{i=1}^{m}\sum_{j=1}^{m}\rho_{ij}v_iv_j+\sum_{j=1}^{m}b_jv_j+\sum_{i=1}^{n}c_ih_i,$$

where ω_{ij}, σ_{ij} and ρ_{ij} represent weights, and b_j and c_i represent biases. Here, we only expand the series up to second-order terms. Let us assume that the income structure among N agents obeys the Boltzmann-like distribution (2). Thus, the joint probability distribution among N agents, $P(\boldsymbol{h},\boldsymbol{v})$, can be derived as (see SI):

$$P(\boldsymbol{h},\boldsymbol{v})=\frac{1}{Z}e^{-\frac{E(\boldsymbol{h},\boldsymbol{v})-N\cdot\mu}{\theta}}, \tag{6}$$

where $Z=\sum_{\boldsymbol{h}}\sum_{\boldsymbol{v}}e^{-\frac{E(\boldsymbol{h},\boldsymbol{v})-N\cdot\mu}{\theta}}$ denotes the partition function.

For simplicity, we consider $\sigma_{ij}=\rho_{ij}=0$. However, our result holds for general case as well. Following the tradition of deep learning, we consider the neural networks consisting of hidden and visible neurons. If we regard agents $\mathbf{h}=(h_1,\dots,h_n)$ as the "hidden neurons" and agents $\mathbf{v}=(v_1,\dots,v_m)$ as the "visible neurons", an N-human society resembles a neural network in which $\boldsymbol{v}$ serves as a signal input, as depicted in Fig. 2a. In such a neural network, it is not important to assign a given social member as being a visible neuron or a hidden neuron. The assignment can be regarded as a random process. The probability that the neural network assigns to the visible vector $\boldsymbol{v}$ is given by summing over all possible hidden vectors: $\sum_{\boldsymbol{h}}P(\boldsymbol{h},\boldsymbol{v})$; therefore, the Boltzmann machine is determined by the maximum likelihood estimate:

$$\underset{\{\omega_{ij}^*,c_i^*,b_j^*\}}{Max}:ln\left(\sum_{\boldsymbol{h}}P(\boldsymbol{h},\boldsymbol{v})\right). \tag{7}$$

The "maximum likelihood" formulation of the optimal problem (7) indicates that the Boltzmann-like distribution (2) "spontaneously" induces the human society to form a Boltzmann machine. That is to say, the observed society is the one that is the most probable among all possible organisms.

5. Three brain-like features

The Boltzmann machine (7) has two properties relating to a brain's learning, which suggests that a human society obeying the Boltzmann-like income distribution (2) resembles a brain. First, the learning rule is local, which makes Boltzmann machine learning biologically plausible [52]. To see this, let us order $(\mathbf{h},\mathbf{v})\in\{0,1\}^{n+m}$. By Eqs. (5) and (6), for a given signal input $\boldsymbol{v}$, the probability of activating the ith hidden neuron can be calculated as [49]:

$$P(h_i=1|\mathbf{v})=\sigma\left(\sum_{j=1}^{m}\omega_{ij}^*v_j+c_i^*\right), \tag{8}$$

where $\sigma(t)=\frac{1}{1+e^{-t}}$. Eq. (8) is the well-known McCulloch-Pitts model of neurons [53]. Second, Boltzmann machines are unsupervised-learning systems, which captures the feature of human-like learning. However, these features of the Boltzmann machine (7) do not establish whether such a human society resembles a real brain enjoying "self-motivation". However, we immediately show that, due to the self-referential Eq. (3), the Boltzmann machine (7) is a self-motivated system, which differs from Hinton's version [46–48]. Using Eqs. (3) and (6), Eq. (7) can be rewritten in the form [54]:

$$\underset{\{\omega_{ij}^*,c_i^*,b_j^*\}}{Max}:ln\left(\sum_{\boldsymbol{h}}P(\boldsymbol{h},\boldsymbol{v})\right),\quad s.t.\ \sum_{\boldsymbol{h}}\sum_{\boldsymbol{v}}P(\boldsymbol{h},\boldsymbol{v})=1 \tag{9}$$

where

$$P(\boldsymbol{h},\boldsymbol{v})=e^{-T} \tag{10}$$

and

$$T=E(\boldsymbol{h},\boldsymbol{v})-N\cdot lnN. \tag{11}$$

Here, $P(\boldsymbol{h},\boldsymbol{v})$ denotes the probability that N agents remain in the state $(\boldsymbol{h},\boldsymbol{v})$, and Eq. (11) is the solution of Eq. (3). Without loss of generality, the coefficient in front of $E(\boldsymbol{h},\boldsymbol{v})$ has been absorbed into Eq. (5). The constraint $\sum_{\boldsymbol{h}}\sum_{\boldsymbol{v}}P(\boldsymbol{h},\boldsymbol{v})=1$ indicates that the human society cannot be of the traditional Boltzmann machine type that has been commonly studied before. To see this, we observe that Eqs. (10) and (11) imply

$$E(0,0)\geq N\cdot\ln N\approx\ln N!>0 \tag{12}$$

for $N\geq2$.

This is a quite surprising result, which means that a neural network always has a positive energy even if each neuron is inactive. It significantly differs from traditional Boltzmann machine, where $E(0,0)=0$. In particular, we again observe that the Gibbs term $\ln N!$ appears in Eq. (12). This leads to the presence of a kind of zero-point energy (usually related to quantum effects in physics), which is consistent with the existence of the Gibbs term in Eq. (4). Due to this positive zero-point energy, the Boltzmann machine (7) or (9) can activate its neurons by itself; that is, the iterative algorithm $E_0\to E_0+E_1\to E_0\dots$ leads to an infinite run process [54]. In this sense, the Boltzmann machine (7) or (9) is a self-motivated system. Furthermore, we point out that the Boltzmann machine (7) or (9) is in accordance with the minimum free-energy principle of the brain theory [55]. To see this, let us write down the free energy of the visible vector $\boldsymbol{v}$:

$$F(\boldsymbol{v})\propto-ln\left(\sum_{\boldsymbol{h}}e^{-E(\boldsymbol{h},\boldsymbol{v})+N\cdot lnN}\right)=-ln\left(\sum_{\boldsymbol{h}}P(\boldsymbol{h},\boldsymbol{v})\right). \tag{13}$$

Obviously, the maximum likelihood procedure (9) is equivalent to minimizing the free energy (13). Therefore, Boltzmann machine (7) or (9) satisfies the minimum free-energy principle [55]. Because

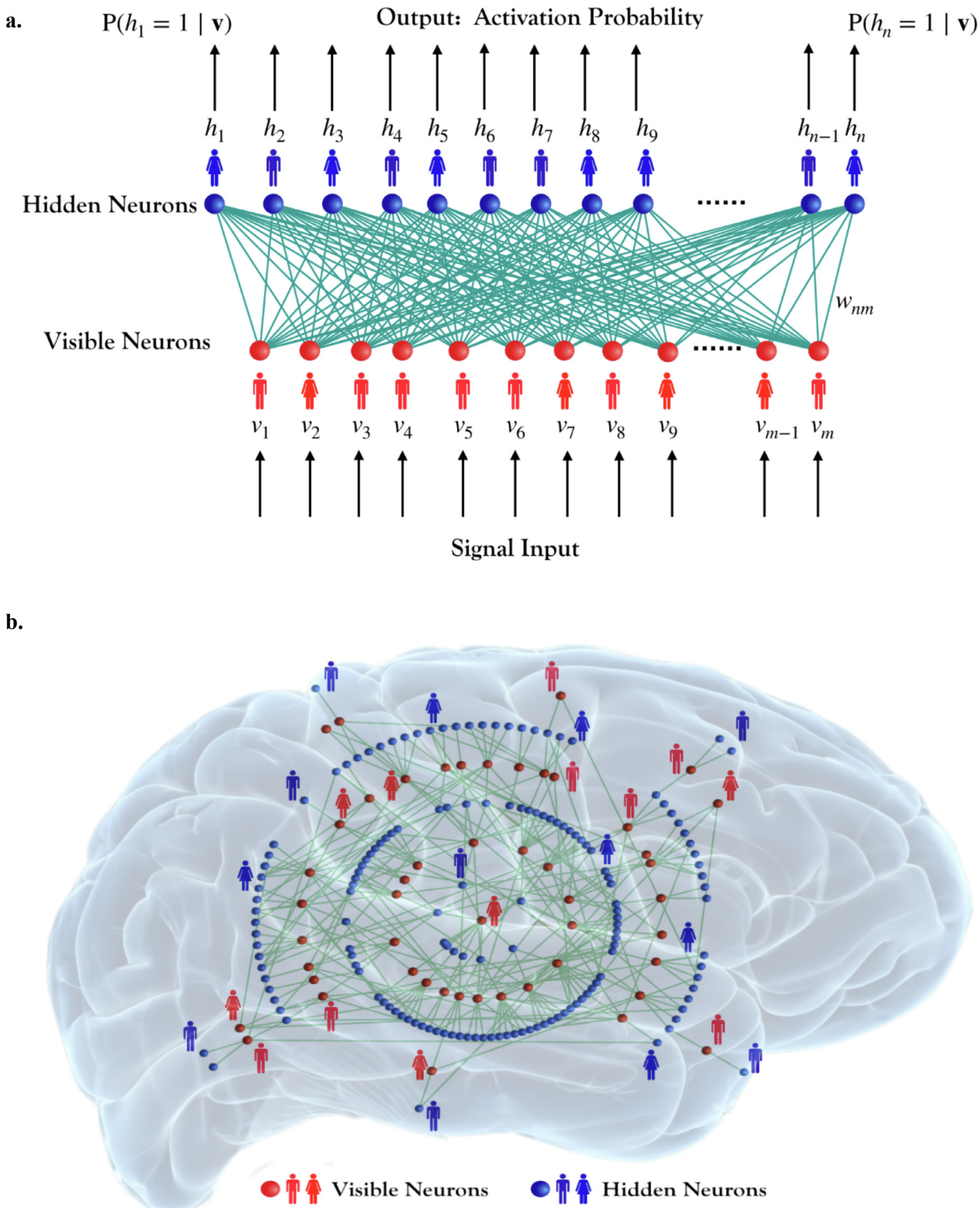

Fig. 2. The well-functioning free-market society spontaneously forms a self-motivated Boltzmann machine. **a:** If a human society obeys the Boltzmann-like income distribution (2), then it can be mapped to a Boltzmann machine consisting of hidden and visible neurons, where each person plays the role of a neuron. **b**: Since a well-functioning free-market society exhibits McCulloch-Pitts learning rule, unsupervised-learning, self-motivation, and satisfies the minimum free-energy principle of the brain theory, it can be thought of a social brain. Here, a pictorial representation of social brain based on bipartite network structure in Fig. 2a is exhibited.

the Boltzmann machine (7) or (9) exhibits McCulloch-Pitts learning rule, unsupervised-learning, self-motivation, and satisfies the minimum free-energy principle of the brain theory, we propose that the Boltzmann-like income distribution (2) is a statistical signature that the human society functions like a social brain as described by Fig. 2b.

Finally, we point out that the Boltzmann machine (9) may have a phase transition that is related to criticality because it is a special type of Ising model. The literature has reported a number of results on collective and swarm intelligence where it seems to appear close to a critical point or, in some system, it emerges as a kind of phase transition [56–61]. In this vein, the Ising model has been considered as an important starting point for understanding the collective behaviors of neurons in a brain [56]. A large literature has reported that the brain operates at or close to criticality [56–58], with mechanisms making criticality attractive to

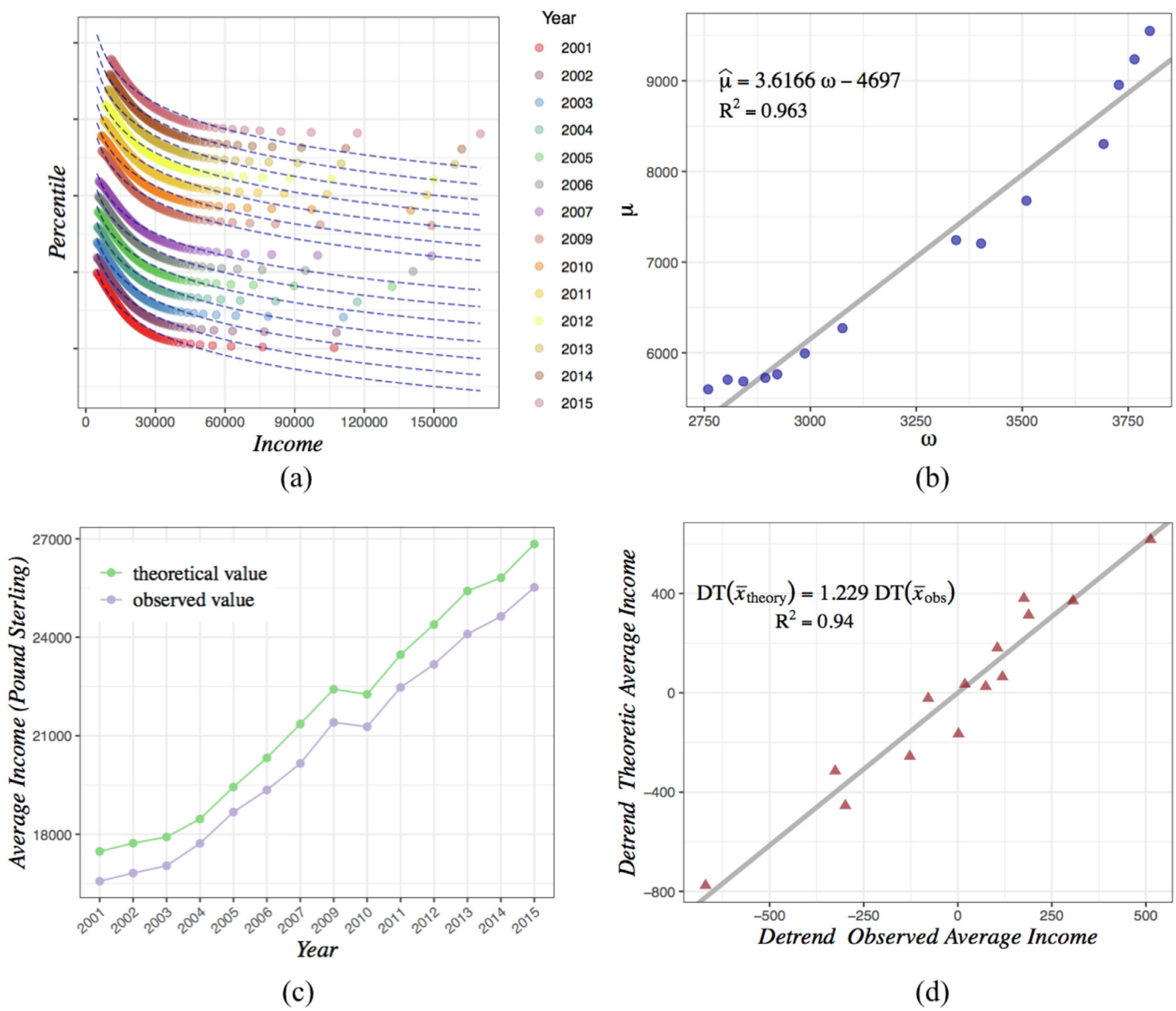

Fig. 3. The test for Boltzmann-like income distribution (2) based on the household data in the United Kingdom**. a**: The household income data from 2001 to 2015 are fitted to the Boltzmann-like income distribution (2), where the fitting results are of good quality except the top income samples (about three quantiles). **b:** The estimate values of μ from 2001 to 2015 are obtained by fitting the Boltzmann-like income distribution (2) to household income data for each year. All estimate values are regressed as a function of the real unemployment compensations. The fitted equation is in accordance with the theoretical result (14). **c:** For the average income for the United Kingdom, the observed value curve is just a translation of the theoretical value curve. Such a translation is simply due to the discreteness of the bins of observed values in the histogram, which are used to compute the average income. **d:** The detrended regression between $\bar{x}_{theory}$ and $\bar{x}_{obs}$ is carried out, where the R^2 still yields 0.94 even when the trend effect has been taken into account.

the dynamics, which is often referred to as “self-organized criticality”. In the picture of the Ising model, the criticality occurs by fine-tuning a control parameter, e.g., the temperature [56]. What could make a fundamentally unstable critical point become self-organized? Sornette et al. have proposed that dynamical feedbacks of the order parameter onto the control parameter provide a general mechanism for this [62–64]. But in general, it is not understood how self-fining to criticality could occur in a living organism. Here we propose that the self-motivated feature of the Boltzmann machine (9) provides a plausible mechanism for achieving the self-fining to criticality. Due to the existence of the zero-point energy, the Boltzmann machine (9) can activate its neurons by itself. Based on this self-motivated mechanism, it is possible for the Boltzmann machine (9) to fine-tune control its parameters by itself (via self-motivated learning). In this sense, the unsupervised-learning function of Boltzmann machine (9) is only a lower intelligence behavior (e.g., mechanical memory). By contrast, we suggest that self-organized criticality could indicate a higher level of intelligence. Indeed, there have been investigations supporting that the consciousness of brain may be related to self-organized criticality [59–61].

6. Empirical evidence for the two-parameter Boltzmann-like distribution

Finally, we carefully investigate if the Boltzmann-like distribution (2) really describes the income structure of human societies. The empirical investigation is divided into two steps. First, we test if the exponential law (2) is in accordance with household income data. Second, we further test if the predicted values of the parameters μ and θ in Eq. (2) agree with actual data. This two-step investigation provides a credible test of the validity of the Boltzmann-like distribution (2).

First, we investigate if the exponential law (2) is in accordance with household income data. Since the Arrow-Debreu general equilibrium model describes a competitive economy with equal oppor-

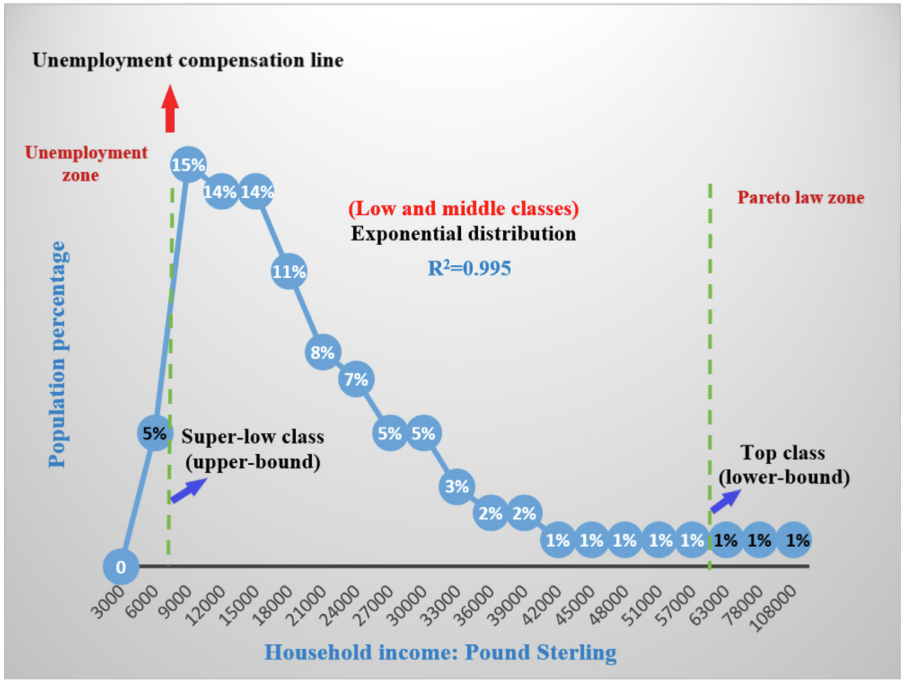

Fig. 4. Income structure of low and middle classes in the United Kingdom (2001) obeys the exponential distribution. The scope of the low and middle income classes ranges from the upper-bound of the super-low income class to the lower-bound of top income class. Both boundaries are marked by two red dotted lines, respectively. The "single peak" in the exponential distribution roughly corresponds to the upper-bound of the super-low income class, which is marked by the "unemployment compensation line".

tunity [24], the Boltzmann-like distribution (2) is expected to describe the low and middle income classes in free-market countries [65]. By contrast, the top income class refers to monopoly and unequal opportunity, and hence obeys the Pareto distribution due to the Matthew effect [26, 65, 66], while the super-low income class refers to unemployment. Both of them fail to satisfy the setting of the Arrow-Debreu general equilibrium model, and hence do not conform to the exponential law (2). Thus, the scope of the low and middle income classes ranges from the upper-bound of the super-low income class to the lower-bound of top income class, see Fig. 4. During the past two decades, there had been many reports confirming the validity of an exponential law in describing the low and middle income classes [30, 65, 67–69]. For example, Tao et al [65] have recently analyzed the datasets of household income from 66 countries and Hong Kong SAR, ranging from Europe to Latin America, North America and Asia. For all these countries, they find that the income distribution for low and middle income classes of populations follows the exponential law (2). As a typical example, in this paper we only investigate the low and middle income classes in the United Kingdom from 2001 to 2015. The fitting results are of good quality, where the adjusted R^2 values are higher than 0.99, see Fig. 3a.

To intuitively see how the low and middle income classes conform to the exponential law (2), we also depict the household income data of United Kingdom in 2001 as the density function in Fig. 4, where the upper-bound of the super-low income class and the lower-bound of top income class are marked by two red dotted lines, respectively. The adjusted R^2 of fitting the exponential law (2) to household income data of United Kingdom in 2001 is 0.995. It might be superficially confused that the Fig. 4 shows a right-skewed distribution with a single peak as described by a Log-Normal function. However, later we will explain that, according to the scope of application of the exponential law (2), the single peak in Fig. 4 should roughly correspond to the upper-bound of the super-low income class, which is marked by the "unemployment compensation line". This implies that one can use the "peak position"[1] in the exponential law (2) to predict the realistic value of unemployment compensation. By fitting the exponential law (2) to household income data of United Kingdom, we have exactly predicted the evolution of the unemployment compensation in the United Kingdom from 2001 to 2015, see Fig. 3b and the upcoming discussion. By contrast, the Log-Normal function has no predictive power.

Second, we investigate if the predicted values of the parameters μ and θ in Eq. (2) agree with actual data. By using Eqs. (1)–(3), μ can be written as:

$$\mu = \sigma \cdot \omega + c, \tag{14}$$

where $\sigma \geq 0$, $c \leq 0$, and ω denotes unemployment compensation. The derivation for Eq. (14) and the test procedure for the linear relationship between μ and ω can be found in the SI. For the years from 2001 to 2015 in the United Kingdom, the test result has been listed in Fig. 3b, where the adjusted R^2 yields 0.96. More importantly, the present fitting result gives $\sigma = 3.62 > 0$ with a p-value $< 10^{-9}$ and $c = -4697 < 0$ with a p-value $< 10^{-4}$, which are perfectly consistent with the theoretical predictions. To test θ, we employ Eq. (2) to obtain the theoretical average income:

$$\bar{x}_{theory} = \mu + \theta - (x_{max} + \theta) e^{\frac{-(x_{max} - \mu)}{\theta}}, \tag{15}$$

where x_{max} denotes the maximum income of low and middle classes. Using the histogram for household income data in the United Kingdom, we can also calculate the observed average income $\bar{x}_{obs}$. If the predicted value of the parameter θ agrees with actual data, we should have $\bar{x}_{theory} = \bar{x}_{obs}$. The empirical investigation in SI shows $\bar{x}_{theory} \approx 1.051 \cdot \bar{x}_{obs}$ with the adjusted R^2 being 0.99. The difference between $\bar{x}_{theory}$ and $\bar{x}_{obs}$ is due to the use of discrete bins in the histogram, resulting in a simple translation of the

[1] Here, the peak position denotes the truncation position of exponential distribution on the side of the low income class, which is marked by the unemployment compensation line, see Fig. 4. Technically, by fitting exponential distribution (2) to household income data, one can determine the peak position.

observed value curve from the theoretical value curve, see Fig 3c. This difference can be decreased by reducing the length of each bin in the histogram. The derivation for Eq. (15) and the empirical investigation procedure can be found in SI. Here, we also clarify that, for the regression between $\bar{x}_{theory}$ and $\bar{x}_{obs}$, the time trend is insignificant, see Fig. 3d, where the trend effect has been removed. Since Eqs. (2), (14) and (15) are all in good agreement with real data, this lends support to the hypothesis that the Boltzmann-like distribution (2) indeed describes the low and middle income structure of free-market countries, which include 66 samples.

Regarding unequal opportunity, we have mentioned that it triggers the Pareto distribution due to the Matthew effect [26, 65, 66]. From this sense, Pareto distribution and Boltzmann-like distribution lie at two extremes among all kinds of social structures, which imply "the law of jungle (unequal opportunity)" and "justice as fairness (equal opportunity)", respectively. Through time, roughly speaking, we humans have explored four social structures, from hunter-gatherer societies, slave-based societies, feudal hierarchical societies, to various types and levels of democratic societies. Today, with the continuous rise of digital interconnection, instantaneous communications, media and blog-based echo chambers, we humans are undergoing a great change within our social networks, further amplified by the rise of artificial intelligence. Due to the emergence of Boltzmann-like income distribution in 66 free-market countries, we conjecture that our world is going through a critical evolution in the form of a kind of brain-like social organism, namely, a self-motivated Boltzmann machine.

7. Conclusion

We have proposed that swarm intelligence in humans can be defined as a phenomenon of quantitative accumulation leading to qualitative transformation. As long as the number of agents in a free-market society with equal opportunity exceeds a threshold value, a Boltzmann-like income distribution (2) will emerge, where the entropy plays the role of swarm intelligence in human societies. Theoretically, we have shown that a human society obeying the Boltzmann-like distribution (2) will spontaneously form a Boltzmann machine, which exhibits the three brain-like features such as McCulloch-Pitts learning rule, unsupervised-learning, and self-motivation, and satisfies the minimum free-energy principle of the brain theory. Empirically, by checking the household income data from 66 countries and Hong Kong SAR, ranging from Europe to Latin America, North America and Asia, we find that, for all of countries, the income structure for low and middle classes (about 95% of the populations) precisely follows the Boltzmann-like distribution (2). Based on the theoretical and empirical research, we conjecture that our world is going through a critical evolution in the form of a kind of brain-like social organism, namely, a self-motivated Boltzmann machine.

Credit author Statement

Yong Tao and Didier Sornette wrote the manuscript. Yong Tao and Li Lin conducted data analysis. Yong Tao prepared the Figs. 1, 4, and the Supplemental Information. Li Lin prepared the Figs. 1, 2, and 3.

Declaration of Competing Interest

The authors declare no competing interests.

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Supplemental Information

Self-reference of the technological level T:

The Boltzmann-like income distribution can be written in the form [18, 54]: $a_i = \frac{1}{e^{\alpha+\beta\varepsilon_i}}$, by which one has:

$$N = N(L, K) = \sum_{i=1}^{n} a_i. \tag{A.1}$$

$$E = E(N(L, K), T) = \sum_{i=1}^{n} a_i \varepsilon_i. \tag{A.2}$$

By using Es. (A.1) and (A.2), we have:

$$\frac{\partial N}{\partial \alpha} = -N, \tag{A.3}$$

$$\frac{\partial N}{\partial \beta} = -E. \tag{A.4}$$

The differential of Eq. (A.4) yields:

$$dE = -d\left(\frac{\partial N}{\partial \beta}\right) = -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta}\frac{\partial N}{\partial \beta} d\beta. \tag{A.5}$$

By Eq. (A.1), we know that N is a function of α and β; therefore, the complete differential of N gives:

$$dN = \frac{\partial N}{\partial \alpha} d\alpha + \frac{\partial N}{\partial \beta} d\beta, \tag{A.6}$$

which leads to:

$$\frac{\partial N}{\partial \beta} d\beta = dN - \frac{\partial N}{\partial \alpha} d\alpha. \tag{A.7}$$

Substituting Eq. (A.7) into (A.5) yields:

$$dE = -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta} dN - \frac{1}{\beta}\frac{\partial N}{\partial \alpha} d\alpha. \tag{A.8}$$

On the other hand, we have:

$$d\left(\alpha \frac{\partial N}{\partial \alpha}\right) = \alpha d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{\partial N}{\partial \alpha} d\alpha, \tag{A.9}$$

which leads to:

$$\frac{\partial N}{\partial \alpha} d\alpha = d\left(\alpha \frac{\partial N}{\partial \alpha}\right) - \alpha d\left(\frac{\partial N}{\partial \alpha}\right). \tag{A.10}$$

Substituting Eq. (A.10) into (A.8) yields:

$$dE = -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta} dN - \frac{1}{\beta} d\left(\alpha \frac{\partial N}{\partial \alpha}\right) + \frac{\alpha}{\beta} d\left(\frac{\partial N}{\partial \alpha}\right)$$
$$= \frac{\alpha}{\beta} d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{1}{\beta} d\left(N - \alpha \frac{\partial N}{\partial \alpha} - \beta \frac{\partial N}{\partial \beta}\right). \tag{A.11}$$

By Eq. (A.3), one can rewrite Eq. (A.11) in the form:

$$dE = -\frac{\alpha}{\beta} dN + \frac{1}{\beta} d\left(N - \alpha \frac{\partial N}{\partial \alpha} - \beta \frac{\partial N}{\partial \beta}\right). \tag{A.12}$$

Moreover, the complete differential of Eq. (A.2) yields:

$$dE = \frac{\partial E}{\partial N} dN + \frac{\partial E}{\partial T} dT. \tag{A.13}$$

By Eqs. (A.12) and (A.13), it is easy to obtain:

$$T = N - \alpha \frac{\partial N}{\partial \alpha} - \beta \frac{\partial N}{\partial \beta}, \tag{A.14}$$

$$\mu = \frac{\partial E}{\partial N} = -\frac{\alpha}{\beta}, \quad (A.15)$$

$$\theta = \frac{\partial E}{\partial T} = \frac{1}{\beta}. \quad (A.16)$$

Here, Eq. (A.14) is the definition for the technological level (or entropy) T [54]. Eqs. (A.14)–(A.16) are consistent if and only if the technological level T is independent of N. Let us now seek the condition for guaranteeing that T is independent of N. Using Eqs. (A.3), (A.4), (A.15) and (A.16), Eq. (A.14) can be rewritten as:

$$T = N - \frac{\mu}{\theta}N + \frac{1}{\theta}E. \quad (A.17)$$

Using Eqs. (A.15) and (A.16), Eq. (A.17) is equivalent to:

$$N\frac{\partial E(N,T)}{\partial N} + (T-N)\frac{\partial E(N,T)}{\partial T} = E(N,T). \quad (A.18)$$

Eq. (A.18) is the condition for guaranteeing that T is independent of N. Remarkably, Eq. (A.18) is solvable [54]. For example, Eq. (11) is obtained by using the solution of Eq. (A.18). To understand the self-reference, we need to observe that, by Eq. (A.18), E is a function of T and N, i.e., $E = E(N,T)$. Substituting $E = E(N,T)$ into Eq. (A.17), which is the definition of technological level [18,54], we obtain:

$$T = N - \frac{\mu}{\theta}N + \frac{1}{\theta}E(N,T), \quad (A.19)$$

where T is defined by T, indicating a self-reference.

Derivation of the joint probability distribution $P(\boldsymbol{h},\boldsymbol{v})$ among N agents

Here, we give the detailed derivation of Eq. (6). Since the income structure of an N-agent society obeys the Boltzmann-like income distribution (2), the joint probability distribution among N agents, $P^*(1,\ldots,N)$, can be written as:

$$P^*(1,\ldots,N) = \prod_{i=1}^{n}(a_i)^{a_i}, \quad (A.20)$$

where a_i and $P^*(1,\ldots,N)$ have been regarded as the unnormalized probabilities.

To obtain Eq. (A.20), we first assume that the income probability of each agent is independent. Since the probability of obtaining ε_i units of income is a_i, by the assumption of independence, the joint probability among a_i agents, each of which obtains ε_i units of income, equals $(a_i)^{a_i} = \overbrace{a_i \cdot a_i \cdots a_i}^{a_i}$. Eq. (A.20) is the result of taking into account all income levels $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$.

Substituting Eq. (2) into (A.20) yields:

$$P^*(1,\ldots,N) = \frac{1}{\prod_{i=1}^{n} e^{\frac{a_i\varepsilon_i - a_i\mu}{\theta}}} = \frac{1}{e^{\frac{\sum_{i=1}^{n} a_i\varepsilon_i - \left(\sum_{i=1}^{n} a_i\right)\cdot\mu}{\theta}}}. \quad (A.21)$$

Finally, we plug Eqs. (A.1) and (A.2) into Eq (A.21) to obtain:

$$P^*(1,\ldots,N) = e^{-\frac{E-N\cdot\mu}{\theta}}. \quad (A.22)$$

Using Eq. (5), $P^*(1,\ldots,N)$ can be written in the form of normalized probability:

$$P(\boldsymbol{h},\boldsymbol{v}) = \frac{1}{Z}e^{-\frac{E(\boldsymbol{h},\boldsymbol{v})-N\cdot\mu}{\theta}}, \quad (A.23)$$

where $Z = \sum_{\boldsymbol{v},\boldsymbol{h}} e^{-\frac{E(\boldsymbol{h},\boldsymbol{v})-N\cdot\mu}{\theta}}$ denotes the partition function.

Derivation for Eq. (14)

By neoclassical economics, the complete differential of Eq. (1) can be specified as [18]:

$$dE = \omega\cdot dL + r\cdot dK + \theta\cdot dT, \quad (A.24)$$

where $\omega = \frac{\partial E}{\partial L}$ denotes the marginal labor return (or unemployment compensation) and $r = \frac{\partial E}{\partial K}$ denotes the marginal capital return (or interest rate).

Moreover, by (A.13) we know that Eq. (2) leads to:

$$dE = \mu dN + \theta dT, \quad (A.25)$$

where $T = N - \alpha\frac{\partial N}{\partial\alpha} - \beta\frac{\partial N}{\partial\beta}$, and N is independent of T if and only if Eq. (A.18) holds. Here we assume Eq. (A.18) holds; therefore, comparing Eqs. (A.24) and (A.25) one has:

$$\mu\cdot dN = \omega\cdot dL + r\cdot dK, \quad (A.26)$$

which can be rewritten as

$$\mu = \sigma\cdot\omega - \sigma\cdot r\cdot MRTS_{LK}, \quad (A.27)$$

where, $\sigma = \frac{dL}{dN}$ denotes the marginal employment level and $MRTS_{LK} = -\frac{dK}{dL}$ denotes the marginal rate of technical substitution of labor and capital. By definition of the marginal employment level, we should have [18,65] $\sigma \geq 0$. Following neoclassical economics, we assume that labor L and capital K are substitute to each other, so we have $MRTS_{LK} \geq 0$. If we observe that the interest rate r is greater than or equal to zero, Eq. (A.27) can be written as:

$$\mu = \sigma\cdot\omega + c,$$

where $\sigma \geq 0$ and $c \leq 0$.

The test procedure for the linear relationship between μ and ω

To do empirical investigation, Eq. (2) can be rewritten in the cumulative distribution form [65]:

$$\begin{cases} F(t \geq x) = e^{\frac{-(x-\mu)}{\theta}} \\ x \geq \mu \end{cases}, \quad (A.28)$$

By fitting Eq. (A.28) to household income data, one can obtain the calibrated values of μ for each country. If the Boltzmann-like income distribution (2) indeed describes actual societies, we should expect that the calibrated value of μ and the unemployment compensation ω announced by governments should obey the linear relationship (14). Since Eq. (a.28) is only suitable for the low and middle income classes, we have to remove both top income data and super-low income data smaller than μ. This leads to a difficulty for the empirical analysis to find the consistent estimate of μ. Regarding this, Tao et al [65] have proved a uniform convergence theorem guaranteeing that the calibration provides a consistent estimate value of μ only by removing the top income part, as long as the sample sizes is large enough. In contrast to other countries, the income data of the United Kingdom contain 99 quantiles (see data resource in Table 1), which are sufficiently large to satisfy the requirement of the uniform convergence theorem. Therefore, we will employ the income data of the United Kingdom to test the validity of Eq. (14). Table 1 has indicated that removing three quantiles in top income samples has guaranteed a quite high adjusted R^2. By fitting the household income data of the United Kingdom to Eq. (A.28), we have computed the estimate value of μ (see Table 2), where we remove only three quantiles in top income samples. Here, we have collected the time series data of unemployment compensation in the United Kingdom from 2001-2015 (except 2008) (see Table 3). Using ordinary least square regressions, fitting the data of Tables 2 and 3 to Eq. (14) yields Fig. 3b. As shown in Fig. 3b, the fitting result is excellent, where the adjusted R^2 yields 0.96. More importantly, the present fitting result gives $\sigma = 3.62 > 0$ with a p-value $< 10^{-9}$ and $c = -4697 < 0$ with a p-value $< 10^{-4}$, which are perfectly consistent with the theoretical predictions in Eq. (14).

Empirical investigation for Eq. (15)

Since the Boltzmann-like distribution (2) is suitable for the low and middle classes, the income interval can be denoted by $\mu \leq x \leq$

Table 1
Adjusted $\boldsymbol{R}^2$ of fitting UK's household income data to exponential distribution.

Year	2001	2002	2003	2004	2005	2006	2007	2009	2010	2011	2012	2013	2014	2015
UK	**0.995**	**0.995**	**0.995**	**0.996**	**0.996**	**0.995**	**0.994**	**0.995**	**0.996**	**0.993**	**0.992**	**0.990**	**0.989**	**0.986**

Data resource: https://www.gov.uk/government/statistics/percentile-points-from-1-to-99-for-total-income-before-and-after-tax
Note: The results of the adjusted R^2 of the fits for the United Kingdom (UK) are listed in Table 1, where, for each year, three quantiles in the top income samples are removed.

Table 2
(Pound sterling): observed values of $\boldsymbol{\mu}$.

Year	2001	2002	2003	2004	2005	2006	2007	2009	2010	2011	2012	2013	2014	2015
μ	**5597**	**5703**	**5684**	**5723**	**5763**	**5993**	**6271**	**7242**	**7204**	**7677**	**8302**	**8953**	**9236**	**9549**

Table 3
(Pound sterling): Unemployment compensation in UK.

Year	2001	2002	2003	2004	2005	2006	2007	2009	2010	2011	2012	2013	2014	2015
ω	**2759**	**2805**	**2842**	**2894**	**2922**	**2987**	**3076**	**3344**	**3403**	**3510**	**3692**	**3728**	**3765**	**3801**

Data resource: https://stats.oecd.org/Index.aspx?DataSetCode=FIXINCLSA

Table 4
(Pound sterling): Observed and theoretical values of average income of households.

Year	$\bar{x}_{obs}$	$\bar{x}_{theory}$
2001	16569	17475
2002	16816	17725
2003	17039	17916
2004	17720	18463
2005	18675	19439
2006	19346	20325
2007	20157	21358
2009	21408	22415
2010	21273	22261
2011	22470	23468
2012	23173	24388
2013	24094	25410
2014	24634	25809
2015	25520	26841

x_{max}, where x_{max} denotes the maximum income of low and middle classes.

Thus, the theoretical average income $\bar{x}_{theory}$ can be computed as:

$$\bar{x}_{theory} = \int_{\mu}^{x_{max}} \frac{1}{\theta} e^{\frac{-(x-\mu)}{\theta}} x dx = \mu + \theta - (x_{max} + \theta) e^{\frac{-(x_{max}-\mu)}{\theta}}, \tag{A.29}$$

where θ is obtained by fitting the Boltzmann-like distribution to real household income data. Therefore, we can test Eq. (A.29) to validate θ.

Because the household income data of the United Kingdom contains 99 quantile: $x_1 < x_2 < \ldots < x_{99}$, we can depict them as a histogram and apply the weighted average to estimate the observed average income $\bar{x}_{obs}$; that is,

$$\bar{x}_{obs} \approx \frac{\frac{x_m + x_{m+1}}{2} + \frac{x_{m+1} + x_{m+2}}{2} + \ldots + \frac{x_{95} + x_{96}}{2}}{99}, \tag{A.30}$$

where $x_m \geq \mu$ and $x_{96} = x_{max}$.

Using the available data for the United Kingdom, we obtain Table 4. If the Boltzmann-like distribution exactly describes the low and middle classes, we should have:

$$\bar{x}_{theory} = \bar{x}_{obs}. \tag{A.31}$$

Using Table 4, we obtain the fitting result:

$$\bar{x}_{theory} = 1.051 \cdot \bar{x}_{obs} - 32.24 \tag{A.32}$$

$$\mathbf{se} = (0.01)\ (207.58)\ \boldsymbol{R}^2 = 0.999$$

$$\mathbf{t} = (105.57)\ (-0.15)$$

$$\mathbf{P\ value} = \left(10^{-19}\right)\ (0.88)$$

Since the p value associated with the intercept −32.24 of Eq. (A.32) is equal to 0.88, it is not significant and we cannot reject the null hypothesis that the intercept is zero. Given the high R^2 value, we conclude that Eq. (A.31) is supported by the data.

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