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Jazz Harmonic Analysis as Optimal Tonality Segmentation

When asked to improvise over the chord changes of a tune, jazz musicians, either by an intuitive or formal process, perform an analysis to obtain a harmonic road map that guides them through the possibilities of what notes and scales to play. The purpose of this harmonic analysis is to discover and understand underlying structures in the chord changes. A notation often used in jazz theory texts (e.g., Nettles and Graf 1997; Jaffe 2009) to represent this harmonic structure identifies a segmentation of the chord changes, the key center (or tonality) of each segment, and the harmonic function of each chord with respect to its key center and other chords. Performing such an analysis is a fundamental step if jazz improvisation is to be simulated by software. It is also an interesting and important problem to be considered on its own for the implementation of jazz compositional and teaching tools. This article presents a formulation of and an algorithm for the harmonic analysis of jazz chord sequences. Some familiarity with jazz harmony or traditional harmony is assumed.

Here is an example of the intended kind of analysis. Consider the chord changes for Miles Davis's *Solar* in Figure 1.

These chord changes will be the input given to the harmonic analysis algorithm. The output of the algorithm is an annotated chord chart, as shown in Figure 2.

Key centers are shown below the bars: The key center of bars 1 and 2 is C minor, that of bars 3–6 is F major, that of bars 7–9 is E♭ major, and so on. Arrows and brackets represent dominant resolutions and related minor seventh chords (i.e., the related IIm7s), respectively (Nettles and Graf 1997). (These will be explained further in the section "Structural Analysis.") Roman numeral chord symbols above the chords indicate their harmonic functions with respect to their key centers. For example, the Dm7♭5 and G7♭9 chords in bar 12 function as IIm7♭5 and V7 chords, respectively, resolving to the root of the key center C minor. Once this analysis is performed, the information it provides can be used by a musician to determine what notes to play over the chord changes. This is the subject of numerous jazz theory texts and method books (e.g., Levine 1995; Pass 1996).

This representation of the result of harmonic analysis has sound theoretical basis and is well understood by jazz musicians. It facilitates direct comparison to analyses performed manually, and allows analysis algorithms to be evaluated objectively and compared with one another, since corpora of analyses of jazz standards are available in the literature (Mehegan 1959, 1962, 1964, 1965; Coker 1987).

The main innovation of the harmonic analysis algorithm presented in this article results from the observation that chord functions and harmonic structures (such as dominant resolutions and related IIm7s) are completely determined by a given segmentation and choice of key centers. Harmonic analysis can therefore be formulated and solved algorithmically as a tonality segmentation problem. This formulation and a solution for it in the form of a dynamic programming algorithm are the subject of this article.

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Related Work

Previous studies related to the subject of this article can be categorized into the following areas: grammar-based and rule-based harmonic analysis of jazz chord sequences, harmonic analysis of tonal music, jazz improvisation systems, and jazz theory.

Grammar-Based and Rule-Based Harmonic Analysis of Jazz Chord Sequences

Formal grammar has been used in the study of jazz chord sequences for some time. Steedman (1984) proposes a context-sensitive grammar as a generative

Figure 1. Chord changes for Solar.

Figure 2. A harmonic analysis of Solar.

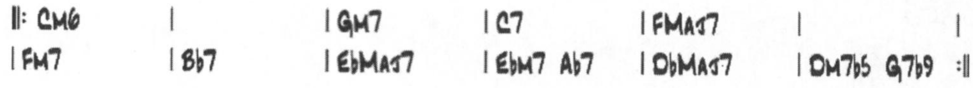


Figure 1.

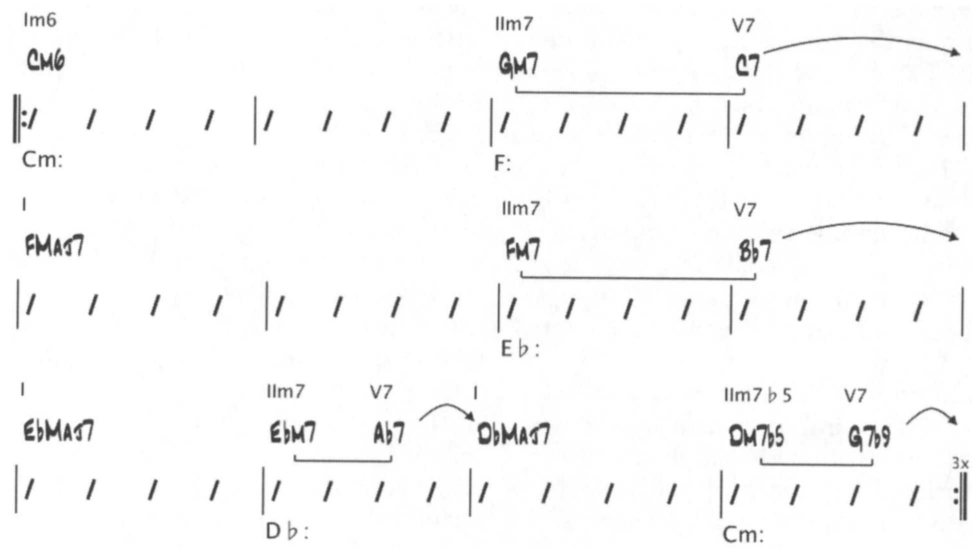


Figure 2.

model of variations of twelve-bar blues changes by chord substitution. The use of formal grammar for harmonic analysis, however, is problematic because no grammar that encompasses all possible chord changes will likely be found. It is also unclear how modulations can be described by grammar rules because "there seem to be no constraints on modulation: a theme can modulate to any new key" (Johnson-Laird 2002, p. 429).

Pachet (1991) proposes a production rule system for harmonic analysis of jazz chord sequences. A rule-based system has the same weakness as a grammar-based system in that it cannot derive a suitable analysis when its set of rules does not include those required to analyze a given tune. For instance, the system is used to determine whether the chord changes of *Solar* are in the form of a blues. Lacking a rule for the general form of these changes, it "produces an analysis which is not what a human would do" (Pachet 2000). Mouton and Pachet (1995) suggest that a symbolic, rule-based method can benefit from a softer decision model by integrating numerical techniques, but offer no concrete proposal as to how this can be applied to harmonic analysis. The algorithm presented in this article incorporates symbolic knowledge in jazz theory and treats tonality segmentation as an optimization problem, and not one of "parsing" the chord sequences. The sample analysis of *Solar* generated by this algorithm (see Figure 2) demonstrates that a chord sequence can indeed be analyzed fully with such a technique. Therefore, with respect to harmonic analysis in the sense taught by jazz theory texts, it is unnecessary to determine whether a tune such as *Solar* is a blues, although it is crucial to find a good tonality segmentation for it.

Scholz, Dantas, and Ramalho (2005) extend Pachet's method by additional processing on gaps: segments of the chord sequence for which no pattern rules apply. After patterns are identified, gaps are merged with neighboring segments when certain conditions are met. Both their and Pachet's methods determine tonality segmentation for a chord sequence by the breaks among adjacent patterns detected by their sets of rules. If tonality segmentation is viewed as an optimization problem, such methods represent greedy algorithms (Corman, Leiserson, and Rivest 2009) in the sense that they make locally optimal choices in hopes that these will lead to globally optimal segmentations. The tonality segmentation algorithm in this article explicitly models the conditions under which modulations may occur and solves the optimization problem directly, using dynamic programming. In the formulation given herein, the quality of the tonality segmentation completely determines that of a harmonic analysis.

Harmonic Analysis of Tonal Music

Much work has also been done in harmonic analysis of tonal music, where the objective is to determine the harmonic structure in a sequence of musical notes. Algorithms proposed for this type of analysis make decisions based on numerical as well as symbolic information. Numerical algorithms for harmonic analysis typically introduce cost and distance functions and formulate the analysis problems as optimization problems. Thus, the algorithm of this article has more in common with them than rule- and grammar-based algorithms for chord sequences. Temperley and Sleator (1999) propose an analysis algorithm based on well-formedness rules and preference rules, adapted from *A Generative Theory of Tonal Music* (Lerdahl and Jackendoff 1983). The well-formedness rules define a solution space while the preference rules define a scoring system for the solutions. Their harmonic analysis problem is then formulated as an optimization problem—one of finding the shortest path in a directed graph, which is solved using dynamic programming.

Pardo and Birmingham (2002) extend Temperley and Sleator's method by also taking chord qualities of segments into consideration. They present results of experiments that evaluate the performance of their algorithm and consider heuristic versions, as well as an optimal version, of the search algorithm.

Illescas, Rizo, and Iñesta (2007) show how chord functions and cadences can be incorporated into preference rules. Stronger cadences are given higher scores, causing analyses that contain them to be preferred. Their technique shows how relationships among consecutive chords and their functions can be reflected in the preference rules. Their harmonic analysis problem is then also formulated as a shortest path problem and solved by dynamic programming.

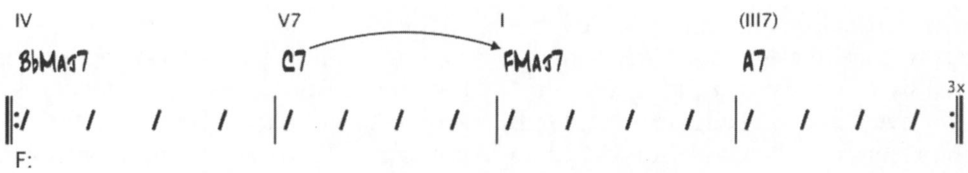
Jazz Improvisation Systems

Numerous research papers and theses have been written about computer-music systems that generate jazz improvisations in various forms (e.g., Ramalho, Rolland, Ganascia 1999; Klein 2005). However, in these studies the problem of harmonic analysis only receives limited attention, and techniques are primarily borrowed from already-existing efforts. Interactive music systems for jazz improvisation apply machine learning techniques to generate improvisation from training data input by users (Pachet 2003; Thom 2003). Without a harmonic analysis component, these systems can only play tunes on which they have been trained, and can only apply or adapt learned lines. Keller and colleagues (2006) design and implement a GUI learning tool that allows its user to "compose an improvisation" by choosing lines to play against each chord (or group of chords) in a chart from a library. It also does not perform harmonic analysis and leaves the decision of which scales to use to its users. All of these systems will benefit from a more complete solution for the problem of harmonic analysis of chord sequences, such as the one presented in this article.

Jazz Theory

The notation for representing harmonic analysis used in the introduction has been taught at least as far back as the 1980s at the Berklee College of Music (Nettles and Ulanowsky 1987) and elsewhere (Jaffe 1983). Not all jazz musicians and teachers agree on

Figure 3. Dominant resolution and deceptive resolution.



the importance of formal harmonic analysis, and there is, of course, an immense body of knowledge on different approaches to jazz improvisation. Most will agree, however, that the study of harmonic analysis is necessary for understanding how and why jazz harmony works. It is also essential for computer simulation of jazz improvisation with explainable and reproducible results.

Jazz theory texts generally describe elements of harmonic analysis by example and define them informally. Specifically, there is no concrete model for modulations, nor is there a systematic procedure for constructing an analysis from a given set of chord changes. A contribution of this article is the formalization of these concepts and descriptions into a mathematical model that can be operated on by computer.

Structural Analysis

The harmonic analysis algorithm operates in two steps: a structural analysis step which converts a chord sequence into a list of analysis elements (defined subsequently) and a tonality segmentation step which partitions this list into segments of different key centers.

In this section the structural analysis algorithm and the data representation for the resulting harmonic structure are described. This description begins with a review of the following elements of jazz theory: dominant resolutions, harmonic rhythm, substitute dominants, related IIm7s, extended dominants, turnarounds, and interpolated dominants (Nettles and Graf 1997; Jaffe 2009).

Review of Jazz Harmony

The *primary dominant* is the V7 chord of a given key. In a major key it is expected to resolve to the IMaj7 chord. (Four-note diatonic chords are prevalent in jazz-related contexts, so in a major key the I chord is usually IMaj7, and in a minor key the Im chord is usually Im7 or ImMaj7. The I6 and Im6 chords can also be used as the I and Im chords, respectively.) For example, in F major, the primary dominant is C7, which resolves to FMaj7. In a major key, the *secondary dominants* are the VI7, VII7, I7, II7, and III7 chords, which are also denoted by V7/II, V7/III, V7/IV, V7/V, and V7/VI, respectively. They are expected to resolve to the diatonic chords IIm7, IIIm7, IVMaj7, V7, and VIm7, respectively. In F major, the secondary dominants are D7 (resolves to Gm7), E7 (resolves to Am7), F7 (resolves to B♭Maj7), G7 (resolves to C7), and A7 (resolves to Dm7).

A primary or secondary dominant resolving to the expected diatonic chord a perfect fifth below creates a *dominant resolution*, which is annotated in a harmonic analysis by a solid arrow (bars 2 and 3 in Figure 3).

A dominant chord that does not resolve to the expected diatonic chord often represents a *deceptive resolution*. A deceptive resolution is not represented by an arrow in a harmonic analysis, but by a parenthesized roman numeral chord (bars 1 and 4 in Figure 3). Note that deceptive resolutions are difficult to represent and analyze using grammar rules.

For simplicity, tunes to be analyzed are assumed to be composed of sections whose lengths in bars are multiples of four, with four beats to a bar. The *metrical structure* (Lerdahl and Jackendoff 1983) of every four bars of a chord sequence is given by the grid in Figure 4. The majority of jazz standards satisfy this assumption. For tunes that do not, other means of deducing the metrical structure must be used, which will not be covered here.

Stronger beats are ones with higher numbers. *Harmonic rhythm* is the pattern of accents created by the chord changes. Certain harmonic elements are

Figure 4. Four-bar metrical structure.

Figure 5. Substitute dominant resolutions and related IIm7s.

Figure 4.

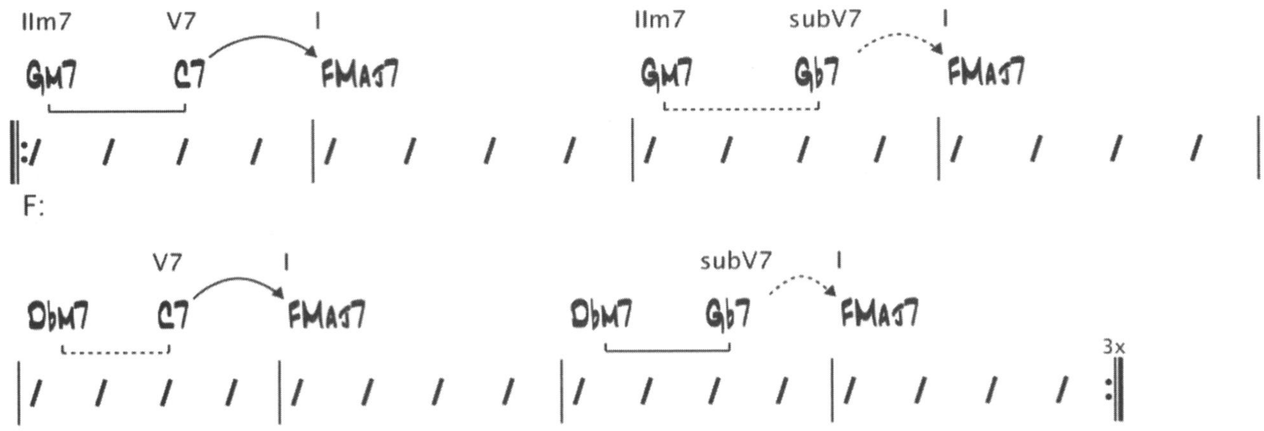


Figure 5.

identified by interaction between harmonic rhythm and metrical structure. A dominant resolution is heard only when a dominant chord on a weaker beat resolves to its target chord on a stronger beat. In Figure 3, the dominant chords C7 and A7 occur on weaker beats compared to the FMaj7 and B♭Maj7 chords into which they resolve.

The *tritone substitution* for a dominant chord is the dominant chord whose root is a ♯IV interval below (or equivalently, a ♭V interval above) the first chord's root. These *substitute dominants* can be used in place of the corresponding primary and secondary dominant chords. In a major key, the substitute dominants are the ♭II7, ♭III7, ♭V7, and ♭VI7 chords, which are also denoted by subV7, subV7/II, subV7/IV, and subV7/V, respectively. The chords IV7 (subV7/III) and ♭VII7 (subV7/VI) will only function as substitute dominants in rare situations (Nettles and Graf 1997). Substitute dominant resolutions are annotated in a harmonic analysis by dotted arrows (G♭7 resolving to FMaj7 in Figure 5).

The *related IIm7* of a dominant chord is the minor seventh chord (or a minor seventh flatted fifth chord for minor tonalities) whose root is a perfect fourth below the dominant chord's root. Any dominant chord may be preceded by its related IIm7, or its tritone substitution's related IIm7. In a harmonic analysis, this is annotated by a solid bracket (or dotted bracket, respectively) under the two chords (see Figure 5). Harmonic rhythm must also be taken into consideration in the detection of related IIm7s: the related IIm7 and the dominant chord must be on a stronger beat and a weaker beat, respectively. (This applies regardless of whether a tritone substitution is used for the dominant chord and whether the IIm7 chord is the related IIm7 of the dominant chord or its tritone substitution.)

An *extended dominant* (also called sequential dominant and substitute sequential dominant) is a series of dominant chords each resolving (deceptively) to the next one. It is represented by a series of arrows in a harmonic analysis (bars 1–4 in

Figure 6. Examples of extended dominants.

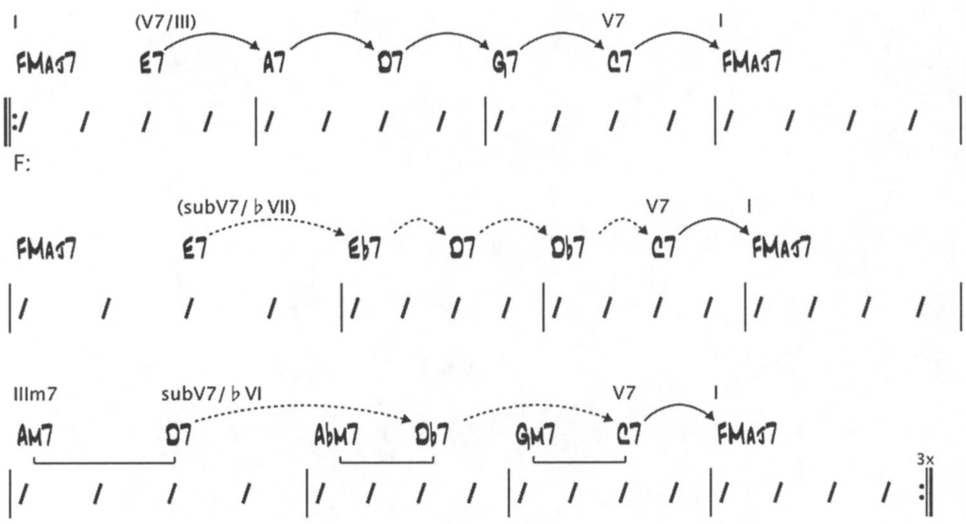


Figure 6). Only the first and last dominant chords are labeled with roman numeral chords. An extended dominant may also be composed of a combination of dominant and substitute dominant resolutions (bars 5–8 in Figure 6). It may also contain related IIm7s (bars 9–12 in Figure 6).

A *turnaround* is an idiomatic progression of (typically) four chords occurring at the end of a section, replacing an extended duration of a tonic chord. Many turnarounds cannot be analyzed by the usual rules. Because their chord progressions are so distinctive, however, the structural analysis algorithm can recognize them by looking up an internal library of turnarounds. An example of a turnaround in F major is the two bars | Am7 A♭m7 | Gm7 G♭7 |. The library of turnarounds used in the current implementation of the structural analysis algorithm contains 13 turnarounds extracted from the collections of tunes in Appendix D of Coker (1987).

An *interpolated dominant* is a substitute dominant chord that is inserted into a larger structure. It always resolves to the chord that follows it, whose root is a ♭II interval below its own root. That target chord must be part of a larger structure. It may be the target of a dominant resolution (the FMaj7 chord in bar 3 in Figure 7). The interpolated dominant is depicted with a straight dotted arrow connecting it to its target. It may be the dominant chord of a related IIm7-dominant chord pair (the C7 chord in bar 6 in Figure 7). Or, it may be the second or subsequent dominant chord in, or the final target of, an extended dominant (the chords G7, C7, and FMaj7 in bars 11, 12, and 1 in Figure 7).

Structural Analysis Algorithm

The structural analysis algorithm and the data structure it generates are now described. Consider the variation of the blues progression in Figure 8.

Figure 9 shows its harmonic analysis. Note that the entire chord sequence is composed of a single segment with a F major key center.

The structural information that needs to be extracted from the chord sequence to generate the harmonic analysis above is represented by boxes in Figure 10.

Each chord, turnaround, interpolated dominant, related IIm7, or extended dominant is represented by an *analysis element* (*AE*). (AEs are representations of basic elements of jazz theory used in harmonic analysis; contrast them with "analysis objects" in Pachet [2000] which represent such elements as well as high-level concepts such as "shapes" and song forms.) The output of the structural analysis algorithm is simply a list of AEs, that is, an *AE list*. Each chord is represented by a *chord AE*. A

Figure 7. Examples of interpolated dominants.

Figure 8. A blues progression.

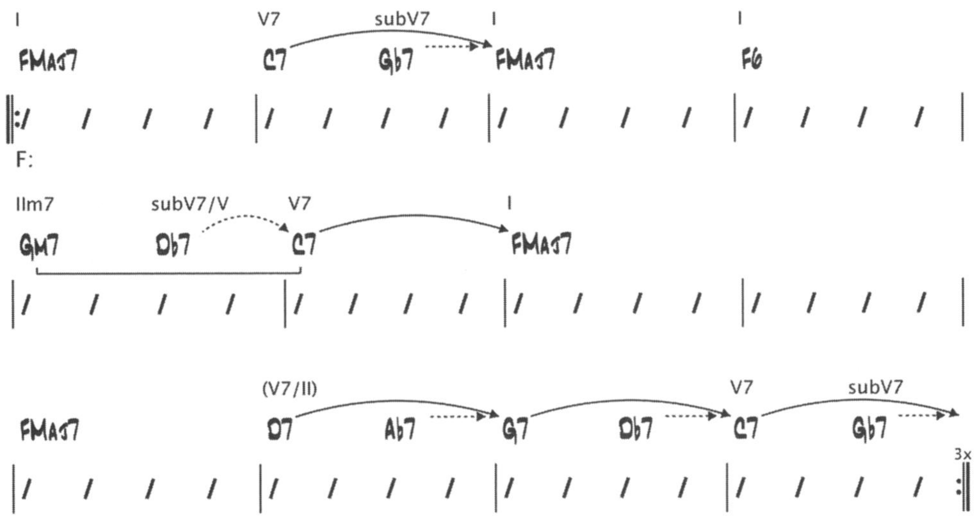


Figure 7.

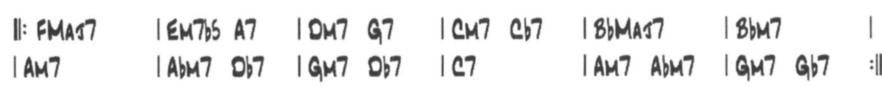


Figure 8.

turnaround AE contains an AE list, the elements of which are chord AEs. An *interpolated dominant AE* contains AEs for the substitute dominant chord and its target chord. A *related IIm7 AE* contains AEs for the IIm7 chord and the "V7"; the latter may be the dominant chord to which the IIm7 chord is related or that dominant chord's tritone substitution, or an interpolated dominant whose target chord is a suitable dominant chord. An *extended dominant AE* contains an AE list; each of its elements may be a chord AE that represents a dominant chord, an interpolated dominant AE, or a related IIm7 AE.

Note that dominant resolutions, substitute dominant resolutions, and deceptive resolutions are not represented explicitly in the data structure. AEs that represent dominant chords, related IIm7s, extended dominants, and turnarounds ending in dominant chords resolve (normally or deceptively) to the AEs that follow them. For example, the AEs corresponding to bars 2 (Em7♭5 A7), 3 (Dm7 G7), and 8 (A♭m7 D♭7) in Figure 9 all function as dominants that resolve to the AEs that follow them, respectively. The AE corresponding to bar 4 (Cm7 C♭7) is a substitute dominant that resolves to the B♭Maj7 chord in bar 5. The related IIm7 (which contains an interpolated dominant) in bars 9 and 10 (Gm7 D♭7 C7) resolves deceptively to Am7, the first chord of the turnaround in bars 11 and 12.

Here is the structural analysis algorithm.

Input: an AE list of input chords.
Output: an AE list representing the result of the analysis.

1. Scan the input AE list for turnarounds; for each one found, replace the chord AEs in it with a turnaround AE.
2. Scan the AE list for related IIm7s (*); for each one found, replace the two AEs forming the related IIm7 with a related IIm7 AE.
3. Scan the AE list for extended dominants (*); for each one found, replace the AEs in it with an extended dominant AE.

Figure 9. A harmonic analysis of the blues progression.

Figure 10. Structural information in the blues progression.

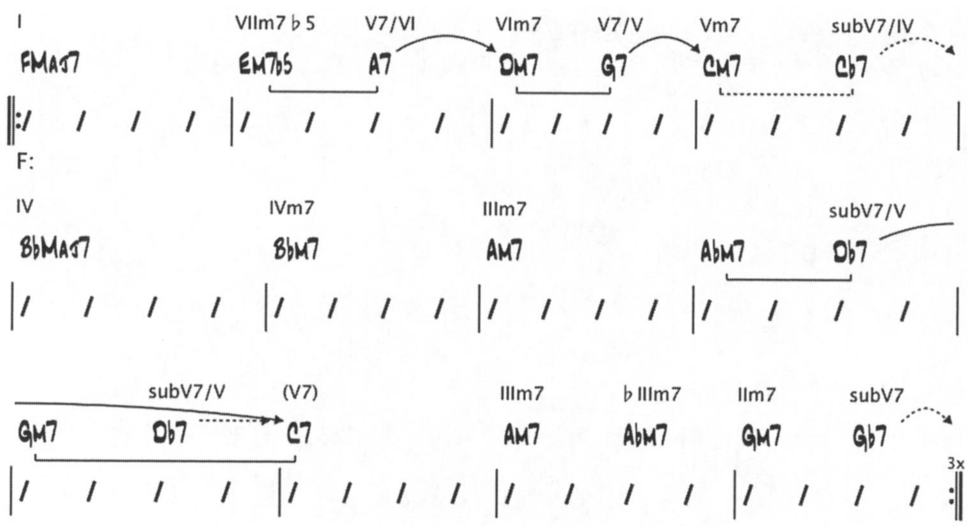


Figure 9.

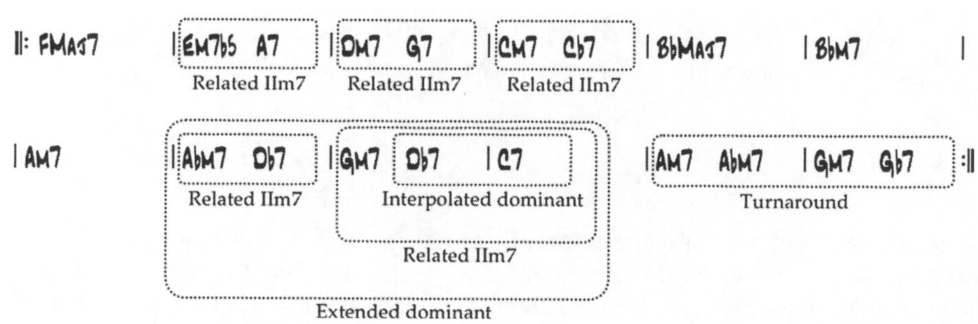


Figure 10.

4. Scan the AE list for interpolated dominants at its top level; for each one found, replace the two AEs in it with an interpolated dominant AE.

(*) When looking for a dominant chord in step 2 and 3, take into account that it may be preceded by an interpolated dominant chord and construct an interpolated.

The four passes of the algorithm detect turnarounds, related IIm7s, extended dominants, and top-level interpolated dominants, respectively. Note that the algorithm cannot scan for interpolated dominants all in one step because they can only be identified when they are part of a larger structure. Clearly the running time of the structure analysis algorithm is a linear function of the number of input chords.

Note also that the structural analysis algorithm does not detect dominant chords in blues progressions and special situations where they do not create a dominant or deceptive resolution. These will be handled by the tonality segmentation algorithm.

The input to the structural analysis algorithm is a list of chord AEs constructed from the input chord sequence. The duration of each chord, measured in beats, is also stored in the chord AE. The following list represents the chord sequence of the sample blues progression.

```
[ (FMaj7, 4); (Em7♭5, 2); (A7, 2); (Dm7, 2); (G7, 2);
  (Cm7, 2); (C♭7, 2); (B♭Maj7, 4); (B♭m7, 4);
  (Am7, 4); (A♭m7, 2); (D♭7, 2); (Gm7, 2); (D♭7, 2);
  (C7, 4); (Am7, 2); (A♭m7, 2); (Gm7, 2); (G♭7, 2);
  (FMaj7, 4)]
```

From this representation the starting beat of each chord can be deduced, which enables us to check harmonic rhythm requirements when locating related IIm7s. Here is the output generated for this input by the structural analysis algorithm.

```
[ FMaj7;
  RelatedIIm7 (Em7♭5, A7);
  RelatedIIm7 (Dm7, G7);
  RelatedIIm7 (Cm7, C♭7);
  B♭Maj7;
  B♭m7;
  Am7;
  ExtendedDominant
  [ RelatedIIm7 (Abm7, D♭7);
    RelatedIIm7
    ( Gm7,
      InterpolatedDominant (D♭7, C7))];
  Turnaround [Am7; A♭m7; Gm7; G♭7];
  FMaj7]
```

Chord AEs are listed by just the names of the chords. In actual implementation each chord's starting beat and duration are stored with the chord to make it easy to check harmonic rhythm requirements for dominant resolutions and generate the harmonic analysis output.

An important observation is that the structural analysis algorithm does not require prior knowledge of the key center to work. Interpolated dominants, related IIm7s, and extended dominants can be identified without knowing the underlying tonality. Also in practice, chord progressions in turnarounds are so distinctive that they can be identified by deducing the key centers from the matched chords. For example the turnaround library contains the pattern "IIIm7 ♭IIIm7 IIm7 ♭II7", which will match "Am7 A♭m7 Gm7 G♭7" in the chord sequence. The presence of the turnaround "Am7 A♭m7 Gm7 G♭7" also implies that the current key center is either F major or F minor. This information will be used to define the cost of a turnaround in the subsequent "Cost Function for Segments" section.

Because of this, the structural analysis algorithm here can also be applied as it is to chord sequences containing tonality changes. The output generated by the structural analysis algorithm for the tune *Solar*, for example, whose analysis is given in Figure 2, is as follows.

```
[ Cm6;
  RelatedIIm7 (Gm7, C7);
  FMaj7;
  RelatedIIm7 (Fm7, B♭7);
  E♭Maj7;
  RelatedIIm7 (E♭m7, A♭7);
  D♭Maj7;
  RelatedIIm7 (Dm7♭5, G7);
  Cm6]
```

The key centers are needed to generate the roman numeral chords in the analysis output. However, this operation is only performed when the tonality segmentation algorithm is complete, at which time the key centers are known.

Tonality Segmentation

Conceptually, the tonality segmentation algorithm is quite simple. It divides the AE list generated by the structural analysis algorithm into segments and assigns a key center to each segment. Its goal is to divide the AE list at positions where modulations occur in the represented chord sequence, and to assign key centers to the segments that "best explain" each chord's harmonic function with respect to the key center of its segment (to be quantified by a cost function below). These two aspects of the algorithm will be discussed in "Validity Conditions for Segments" and "Cost Function for Segments," respectively. First, the use of dynamic programming for tonality segmentation is described.

A Dynamic Programming Algorithm for Tonality Segmentation

Let $a_0, a_1, \ldots, a_{n-1}$ be the AE list output of the structural analysis algorithm. Each a_i, $0 \le i < n$, is a top-level AE on the AE list. For example, for the AE list for *Solar*, $n = 9$, a_0 is the chord AE for Cm6, a_1 is the related IIm7 AE for Gm7 and C7, a_2 is the chord AE for FMaj7, and so on.

A segmentation of the AE list into m segments is represented by indexes $p_0, p_1, \ldots, p_m$ such that $0 = p_0 < p_1 < \cdots < p_m = n$. The i-th segment contains the AEs $a_{p_i}, a_{p_i+1}, \ldots, a_{p_{i+1}-1}$, for $0 \le i < m$. The objective of the algorithm is to find a segmentation with minimal cost, where the cost of a segmentation is defined as follows. Let $d^k_{i,j}$ be the cost (described later) of assigning the key center k to the segment $a_i, a_{i+1}, \ldots, a_j$, where $0 \le i \le j < n$, $k \in K$, and K is the set of all possible major and minor keys. Let $d^*_{i,j}$ be the minimal cost of $d^k_{i,j}$ among all keys, i.e.,

$$d^*_{i,j} = \min_{k \in K} d^k_{i,j}$$

Then the cost of the segmentation $p_0, p_1, \ldots, p_m$ is given by

$$M(m-1) + \sum_{i=0}^{m-1} d^*_{p_i, p_{i+1}-1}$$

where M is a constant that represents the cost of a modulation. The choice of its value will be discussed subsequently.

A minimal-cost segmentation can be found using dynamic programming. Let c_i be the minimal cost for segmenting $a_0, a_1, \ldots, a_i$, for $0 \le i < n$. Then,

$$c_i = \begin{cases} d^*_{0,0} & \text{if } i = 0 \\ \min\left(d^*_{0,i}, \min_{j=1}^{i}\left(c_{j-1} + d^*_{j,i} + M\right)\right) & \text{if } i > 0 \end{cases}$$

That is, apart from the initial condition, the minimal cost of analyzing the first i AEs is given by either the cost of analyzing all of them in one key center, or the minimum of the sum of the minimal cost of analyzing the first j AEs, that of analyzing the remaining $i - j$ AEs in one key center, and the cost of a modulation, over all possible values of j—whichever is smaller. Given the values of $d^k_{i,j}$, for $0 \le i \le j < n$ and $k \in K$, the values of $d^*_{i,j}$ for $0 \le i \le j < n$ can be determined in $n^2\,|K|$ operations. Because $|K|$ is constant (24, if 12 major keys and 12 minor keys are considered), this step takes $O(n^2)$ time. The values of c_i for $0 \le i < n$ can also be computed in $O(n^2)$ time in the order of increasing index i. A standard technique in dynamic programming is used to record the index j that results in the minimal cost in each step so that the minimal segmentation can be recovered after c_{n-1} has been computed.

This formulation of tonality segmentation makes the assumptions that changes in tonality do not occur within the AEs and that the tonality of each segment is independent of those of past and future segments. In practice these assumptions do not hinder the discovery of the "correct" segmentation. The structural analysis algorithm can thus be viewed as a normalization step performed on the input chord sequences so that the tonality segmentation algorithm can process them more easily.

Cost Function for Segments

To complete the description of the tonality segmentation algorithm, $d^k_{i,j}$, the cost for choosing k as the key center for the segment $a_i, a_{i+1}, \ldots, a_j$ needs to be defined. For example, one would expect $d^{\text{"F major"}}_{1,2}$ to have a small value for the AE list in the given example, because the segment a_1, a_2 (Gm7 C7, FMaj7) corresponds to the most common cadence in F major. Let

$$d^k_{i,j} = \begin{cases} t^k_{i,j} & \text{if } s^k_{i,j} \text{ is true} \\ \infty & \text{otherwise} \end{cases}$$

for all $0 \le i \le j < n$ and $k \in K$. The quantities $s^k_{i,j}$ and $t^k_{i,j}$ play the roles that well-formedness rules and preference rules do in Temperley and Sleator (1999), respectively. That is, $s^k_{i,j}$ defines a space of all valid solutions, and $t^k_{i,j}$ assigns costs to these solutions to reflect their respective quality; the optimization problem is then one of finding a valid solution with minimal cost. The cost measure $t^k_{i,j}$ is defined to be the sum of costs associated with the harmonic

Table 1. Chord Categories and Costs

Category	*Major Key Chords*	*Minor Key Chords*	*Cost*
Root	IMaj7	Im7	–6
Diatonic excluding root and primary dominant	IIm7, IIIm7, IVMaj7, VIm7, VIIm7♭5	IIm7, ♭IIIMaj7, IVm7, Vm7, ♭VIMaj7, ♭VII7	–5
Primary dominant	V7	V7	–4
Substitute primary dominant	♭II7	♭II7	–3
Secondary dominant	VI7, VII7, I7, II7, III7	VI7, ♭VII7, I7, II7, ♭III7, IV7	–2
Substitute secondary dominant	♭III7, ♭V7, ♭VI7, IV7, ♭VII7	♭II7, II7, ♭V7, ♭VI7, VII7	–1
Blues	IV7	—	0
Modal interchange	Im7, IIm7♭5, ♭IIIMaj7, IVm7, Vm7, ♭VIMaj7, ♭VII7	—	0
Diminished	Any diminished chord	Any diminished chord	0
Unknown	Chords not listed above	Chords not listed above	∞

functions of each of the AEs $a_i, a_{i+1}, \ldots, a_j$ with respect to the key center k according to Table 1.

For example, a_1 in the given example, a related IIm7 with chords Gm7 and C7, functions as a primary dominant in F major (as explained subsequently), and contributes a cost of –4 to segments that contain it. The cost assigned to each category indicates how much the presence of an AE in that category supports the choice of the key center. That is, the presence of a root is better evidence in support of the key center than a "diatonic" chord (as defined in Table 1, which excludes the root and the primary dominant), which is in turn better evidence than a primary dominant, and so on. Note that minor keys do not have blues or modal interchange chords, indicated by empty entries in Table 1.

These cost values are chosen by trial and error through experiments conducted on the collection of tunes in Appendix D of Coker (1987). They follow our intuition on the degrees to which chords in these different categories reflect the presence of a given key center. The tonality segmentation algorithm appears to be robust with respect to variations in the cost values as long as relative rankings among the categories are preserved.

To determine the category of a chord AE, its roman numeral chord with respect to k is computed and looked up in the appropriate column of Table 1. The category of an interpolated dominant AE is that of its target chord. The category of a related IIm7 AE is that of its "V7" AE. The category of an extended dominant AE is that of its last AE. The cost of a turnaround depends on the key centers it implies, which are determined when it is detected by the structural analysis algorithm. Its cost is 0 if k is one of those key centers and ∞ otherwise. For example, the cost of the turnaround "Am7 A♭m7 Gm7 G♭7" is 0 if k is F major or F minor, and ∞ otherwise.

The values of $t^k_{i,j}$, $0 \le i \le j < n$, and $k \in K$ can be computed in $O(n^2)$ time:

$$t^k_{i,j} = \begin{cases} \text{category cost of } a_i \text{ from table 1} & \text{if } i = j \\ t^k_{i,j-1} + t^k_{j,j} & \text{otherwise} \end{cases}$$

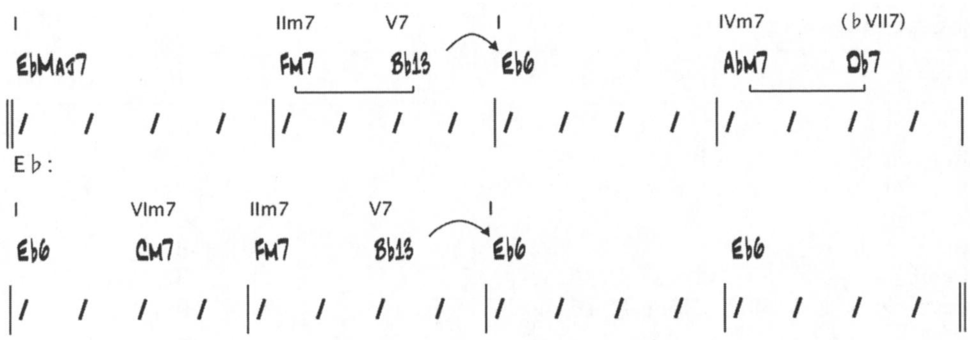
The cost of a modulation, M, is assigned a value larger than that of any segment. That is, let $M > -t^k_{0,n-1}$ for all $k \in K$. Because of this, the tonality segmentation algorithm will always minimize the number of modulations (under the constraint of the validity conditions to be discussed) as well as find an assignment of key centers to the segments that correspond to the best analysis of harmonic functions of all the AEs.

Validity Conditions for Segments

To complete the definition of $d^k_{i,j}$, define $s^k_{i,j}$ to be the validity of analyzing the segment $a_i, a_{i+1}, \ldots, a_j$ in key center k. It is defined as a logical conjunction of four conditions:

$$s^k_{i,j} = r^k_{i,j} \wedge f^k_{i,j} \wedge u^k_{i,j} \wedge e^k_{i,j}$$

Figure 11. Bridge of tune 31a.



A valid segment is assumed to always contain a root AE of key center k. Let $r^k_{i,j}$ be true if and only if the segment $a_i, a_{i+1}, \ldots, a_j$ contains a root AE in key k.

Also, a valid segment is assumed not to end in a dominant chord. Let $f^k_{i,j}$ be true if and only if a_j is not an AE of a category in DOM in key k, where $DOM = \{$Primary dominant, Secondary dominant, Substitute primary dominant, Substitute secondary dominant$\}$.

Each AE in the segment must have a known harmonic function in key k. Let $u^k_{i,j}$ be true if and only if all of $a_i, a_{i+1}, \ldots, a_j$ have known categories in Table 1 with respect to key k.

It is easy to verify that each of $r^k_{i,j}$, $f^k_{i,j}$, and $u^k_{i,j}$ can be computed in $O(n^2)$ time for all $0 \le i \le j < n$ and $k \in K$.

Validity of a Segment Due to Subsegments

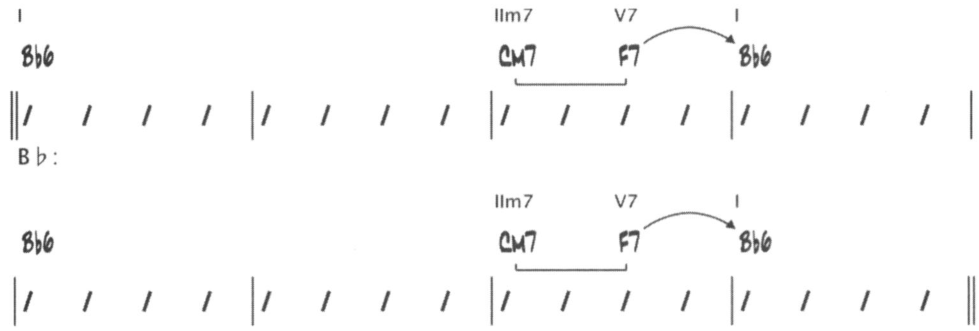
The final condition $e^k_{i,j}$ causes the validity of analyzing the segment $a_i, a_{i+1}, \ldots, a_j$ in key k to be affected by whether subsegments embedded in it can be analyzed as modulations to keys related to k. In other words, $e^k_{i,j}$ is true if and only if the segment $a_i, a_{i+1}, \ldots, a_j$ contains no modulation to a related key that prevents it from being analyzed completely in key k. A key is related to key k if the former's tonic chord has one of the harmonic functions in k listed in Table 1. A related key is specified by a roman numeral interval optionally followed by the letter 'm' (for minor keys). For example, the related keys ♭VI and IIIm of F major are D♭ major and A minor, respectively. Using this notation, the related keys of a major key are IIm, IIIm, IV, VIm, Im, ♭III, IVm, Vm, and ♭VI and those of a minor key are IIm, ♭III, IVm, Vm, and ♭VI.

Modulations to keys unrelated to k are already detected by the use of $u^k_{i,j}$. For example, in the analysis of *Solar* in Figure 2, analysis of the segment containing the AEs a_1 (Gm7 C7), a_2 (FMaj7), and a_3 (Fm7 B♭7) in F major cannot extend into a_4 (E♭Maj7) because E♭Maj7 has no valid harmonic function in F major. Thus, the value of $u^{\text{“F major”}}_{1,4}$ is false. Note also that because a_3 functions as a substitute secondary dominant in F major and as a primary dominant in E♭ major, the cost assignments in Table 1 will associate it with the latter key center after the modulation from F major to E♭ major is detected.

As an example of a tune with a modulation to a related key, consider tune 31a in Appendix D of Coker (1987). This tune begins with an A section with a key center of B♭ major, played twice. It then modulates to E♭ major in the bridge, as shown in Figure 11. This is followed by a repeat of the A section (in B♭ major). The bridge contains a turnaround "E♭6 Cm7 Fm7 B♭13" that implies an E♭ major tonality. Because this turnaround does not have a valid harmonic function in B♭ major, the modulation to E♭ major in the bridge is detected in the same way a modulation to an unrelated key is detected, as described above, using $u^k_{i,j}$.

Now consider tune 31b in Appendix D of Coker (1987). It begins with an A section in F major, played twice. It then modulates to B♭ major in the bridge, shown in Figure 12. This is followed by a variation of the A section (also in F major). The bridge is composed of only two types of AEs,

Figure 12. Bridge of tune 31b.



chord AE B♭6 and related IIm7 AE Cm7 F7. Because these AEs have valid harmonic functions in F major, using only the mechanisms described so far, the bridge will also be analyzed in F major, leaving the modulation to B♭ major undetected. The use of $e^k_{i,j}$ enables the tonality segmentation algorithm to detect modulations within segments even when these modulations do not contain AEs that distinguish them from the tonal centers of the enclosing segments.

The design of $e^k_{i,j}$ presents a challenge because if the condition is too selective, the algorithm will miss subsegments that are in fact modulations to related keys; if it is too indiscriminate, the algorithm will detect superfluous modulations. Therefore, $e^k_{i,j}$ is defined in such a way that allows experimentation and fine tuning. The chord charts with tonality segmentation in Appendix D of Coker (1987) are then used in experiments to determine its final definition, which is presented herein. A future extension to the tonality segmentation algorithm can include a statistical model for modulations and select parameters for defining $e^k_{i,j}$ automatically using a set of training data.

Let $g^k_{i,j}(rk, sc, st, sp)$, for $0 \leq i \leq j < n$ and $k \in K$, be the validity of analyzing segment $a_i, a_{i+1}, \ldots, a_j$ in the key k with respect to whether it contains subsegments that can be identified as modulations according to a criterion specified by the tuple (rk, sc, st, sp). That is, $g^k_{i,j}(rk, sc, st, sp)$ is true if and only if no subsegment of a certain type (specified by (rk, sc, st, sp)) occurs within it, which would invalidate the analysis of the entire segment in k. The first parameter rk is a related key of k. The precise definitions of sc, st, and sp will be given herein. Intuitively, the parameters sc and st provide a test for determining whether a subsegment has the characteristics of a modulation (e.g., being long enough, or containing AEs of the right categories). The parameter sp specifies the allowable positions of a subsegment in $a_i, a_{i+1}, \ldots, a_j$. The complete list of values for (rk, sc, st, sp) that the tonality segmentation algorithm considers is given in Table 2. Then $e^k_{i,j}$ is simply the logical conjunction of the values of $g^k_{i,j}(rk, sc, st, sp)$ over all these sets of values.

Certain related keys such as IV and VIm are "more related" to the original major key than others. Subsegments in these related keys need to be more prominent before they are identified as modulations. Under certain conditions (see "type C," subsequently), these subsegments must be eight bars or longer to be identified as modulations. Subsegments in "less related" keys, such as ♭III and ♭VI, are identified as modulations more easily. For example, in a segment in the key of F major, a short two-bar subsegment of, say, an E♭7 chord followed by an A♭Maj7 chord, is already identified as a modulation.

According to how easily subsegments analyzed in them are identified as modulations, related keys are divided into three types as follows.

- Type A – ♭III and ♭VI of a major or minor key
- Type B – Im, IIm, IIIm, IVm, and Vm of a major key; IIm and Vm of a minor key
- Type C – IV, and VIm of a major key; IVm of a minor key

Table 2. List of All Related Keys, Subsegment Categories, Subsegment Tests, and Subsegment Positions Considered by the Tonality Segmentation Algorithm

		For major key k		
Related Key (rk)	*Subsegment Categories(sc)*	*Subsegment Test (st)*	*Subsegment Position(sp)*	*Type*
♭III	sc_{simple}	st_{simple}	*ANY*	A
♭VI	sc_{simple}	st_{simple}	*ANY*	A
Im	sc_{simple}	st_{simple}	*EP*	B
IIm	sc_{simple}	st_{simple}	*EP*	B
IIIm	sc_{simple}	st_{simple}	*EP*	B
IVm	sc_{simple}	st_{simple}	*EP*	B
Vm	sc_{simple}	st_{simple}	*EP*	B
IV	sc_{long}	st_{long}	*ANY*	C
IV	sc_{short}	st_{short}	*EP*	C
VIm	sc_{long}	st_{long}	*ANY*	C
VIm	sc_{short}	st_{short}	*EP*	C
		For minor key k		
Related Key (rk)	*Subsegment Categories(sc)*	*Subsegment Test (st)*	*Subsegment Position (sp)*	*Type*
♭III	sc_{simple}	st_{simple}	*ANY*	A
♭VI	sc_{simple}	st_{simple}	*ANY*	A
IIm	sc_{simple}	st_{simple}	*EP*	B
Vm	sc_{simple}	st_{simple}	*EP*	B
IVm	sc_{long}	st_{long}	*ANY*	C
IVm	sc_{short}	st_{short}	*EP*	C

These types have increasingly stringent requirements for subsegments analyzed in their related keys to be identified as modulations. Each type has its own criterion (or criteria) for identifying modulations.

A subsegment in a related key of type A is detected by setting the parameters *sc*, *st*, and *sp* to

$$sc = sc_{simple} = \{\text{Root, Primary dominant}\},$$

$$st = st_{simple} = dur(h, l) \geq 8, \text{ and}$$

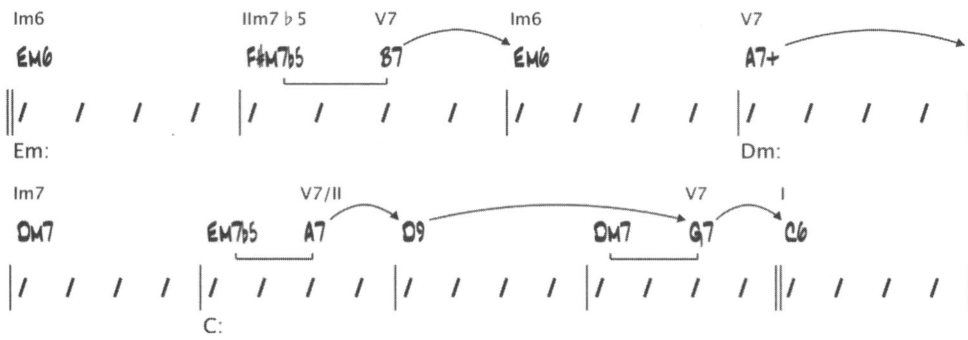
$$sp = ANY$$

respectively, where $dur(h, l)$ denotes the total duration (in number of quarter notes) of the AEs in the subsegment $a_h, a_{h+1}, \ldots, a_l$. In other words, a passage in a related key of type A must be at least two bars long to be considered a modulation. The subsegment $a_h, a_{h+1}, \ldots, a_l$ is identified as a modulation if, when analyzed in the related key *rk*, each AE in $a_h, a_{h+1}, \ldots, a_l$ is of a category in *sc*, the test *st* evaluates to true, and (1) at least one AE of the subsegment is in the Root category and (2) the subsegment does not end in an AE of a category in *DOM*. Conditions (1) and (2) are always added regardless of the values of *rk*, *sc*, *st*, and *sp*. For example, in the key of F major, the subsegment |E♭7|A♭Maj7| is identified as a modulation in the related key ♭III. So are |A♭Maj7|A♭Maj7| and |F7|B♭7|E♭7|A♭Maj7|, but not |Bm♭7|E♭7| (violates condition [1]), |A♭Maj7|E♭7| (violates condition [2]), or |B♭♭7|A♭Maj7| (category of B♭♭7 not in *sc*).

The value *ANY* for the parameter *sp* means that the subsegment $a_h, a_{h+1}, \ldots, a_l$ may appear anywhere within the segment in question, and therefore represents a null test.

Related keys of type B are "more related" to the original key *k* than those of type A. Subsegments in them are identified as modulations when they are immediately preceded or followed by a modulation to yet another key center. They are not identified as modulations when they appear in the middle of a segment in key *k*, however. As an example of modulations to related keys of type B, consider the analysis

Figure 13. Bridge of Yardbird Suite.



of the bridge of tune 73 (*Yardbird Suite*) in Appendix D of Coker (1987), shown in Figure 13. Its first three bars form a segment with a key center of E minor. Its last three bars (and the A section that follows, only the first bar of which is shown) form a segment with a key center of C major. Although bars 4 and 5 can be analyzed in C major, they satisfy the criterion specified below (including being preceded by a segment with another key center, E minor). They are therefore detected as a separate segment with a key center of D minor, a related key of C major of type B. Note also that if the Em6 chords in bars 1 and 3 (no harmonic function in C major) are changed to Em7 chords (diatonic chords in C major), the entire bridge will be analyzed in C major because bars 4 and 5 occur in the middle of a segment with a key center of C major.

Modulations to related keys of type B are detected by setting the parameters *sc*, *st*, and *sp* to

$$sc = sc_{simple},$$
$$st = st_{simple}, \text{ and}$$
$$sp = EP$$

The value of EP for parameter *sp* specifies that the subsegment $a_h, a_{h+1}, \ldots, a_l$ must appear at an endpoint of the segment $a_i, a_{i+1}, \ldots, a_j$. That is, an additional test is performed which evaluates to true if and only if $h = i \vee l = j$.

Chords in the related keys of type C are most related to *k*. Two sets of criteria are needed to correctly handle the different ways in which modulations in these related keys may appear. Longer subsegments in these related keys are identified as modulations regardless of where they are in the original segment. Such subsegments will contain AEs belonging to more categories. The settings for *sc*, *st*, and *sp* to identify these modulations are

$$sc = sc_{long} = \{\text{Root, Diatonic}\} \cup DOM$$
$$st = st_{long} = dur(h,l) \geq 32 \wedge 4 \cdot rdur(h,l) \geq odur(h,l), \text{ and}$$
$$sp = ANY$$

where $rdur(h,l)$ and $odur(h,l)$ denote the total durations (in number of quarter notes) of root AEs and other AEs in the subsegment $a_h, a_{h+1}, \ldots, a_l$, respectively. The modulation from F major in the A section to B♭ major in the bridge in tune 31b in Appendix D of Coker (1987; see Figure 12) is detected by these parameter settings.

Shorter subsegments in related keys of type C are also detected as modulations when they are immediately preceded or followed by a modulation to another key center. The corresponding settings for *sc*, *st*, and *sp* are

$$sc = sc_{short} = \{\text{Root, Primary dominant}\},$$
$$st = st_{short} = dur(h,l) \geq 8 \wedge 2 \cdot rdur(h,l) \geq odur(h,l) \wedge odur(h,l) > 0, \text{ and}$$
$$sp = EP$$

As an example, tune 33 (*Jeepers Creepers*) in Appendix D of Coker (1987) begins with two repeats of an A section in the key of E♭ major, followed by the bridge shown in Figure 14, then followed by another time through the A section. Bar 5 and the first half of bar 6 of the bridge are analyzed in B♭ major because B♭Maj7 has no valid harmonic function in E♭ major. The first four bars can be analyzed in E♭ major as

Figure 14. Bridge of Jeepers Creepers.

Figure 15. Chord changes for Solar *represented by key centers and roman numeral chords.*

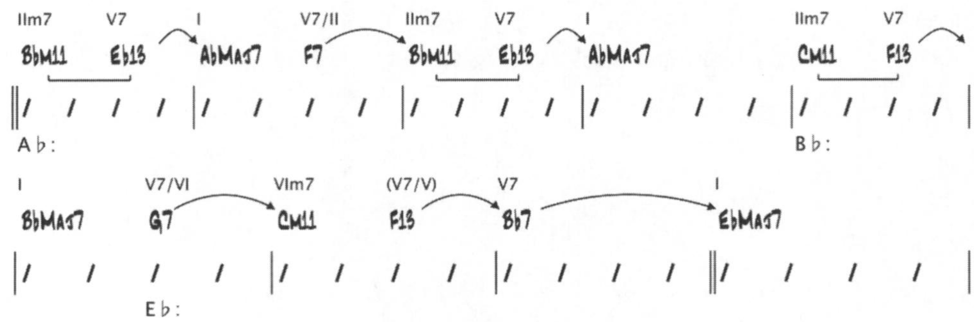


Figure 14.

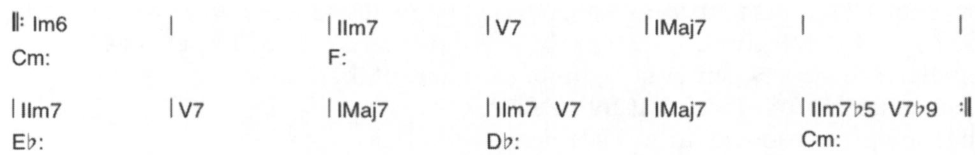


Figure 15.

a continuation of the segment containing the A section. However, they are not, because they are detected by the settings for *sc*, *st*, and *sp* above and analyzed as a separate segment with a key center of A♭ major.

It can be shown that the computation of $e^k_{i,j}$ for all $0 \leq i \leq j < n$ and $k \in K$ requires $O(n^2)$ time. Given k, for each combination of rk, sc, st, and sp, categorize the AEs $a_0, a_1, \ldots, a_{n-1}$ with respect to the related key rk. Identify all maximal spans in it for which: each AE is of a category in sc, st is true, at least one AE in the Root category is present, and the category of the last AE is not in DOM. Intuitively, a segment $a_i, a_{i+1}, \ldots, a_j$ can only be valid for $sp = ANY$ if i and j are in the same maximal span or gap between maximal spans, since any proper subsegment it contains that overlaps a maximal span will invalidate it. (A proper subsegment of a segment is one that is [strictly] shorter than the segment.) Additionally, the segment $a_i, a_{i+1}, \ldots, a_j$ can only be valid for $sp = EP$ if i and j are in the same maximal span or gap between maximal spans, or each of them is in a gap between maximal spans. This ensures that no proper subsegment at its two endpoints overlaps a maximal span, which will invalidate it. Therefore let $z_0, z_1, \ldots, z_{n-1}$ be integer labels assigned to the AEs such that two AEs have the same odd (or even) label if and only if they belong to the same span (respectively, gaps between spans). Then it can be shown that

$$g^k_{i,j}(rk, sc, st, sp) = \begin{cases} z_i = z_j & \text{if } sp = ANY \\ z_i = z_j \vee (z_i \bmod 2 = 0 \wedge z_j \bmod 2 = 0) & \text{if } sp = EP \end{cases}$$

This completes the description of the tonality segmentation algorithm. Note that the time complexity of the harmonic analysis algorithm is $O(n^2)$.

Evaluation of the Tonality Segmentation Algorithm

The performance of the tonality segmentation algorithm is evaluated using the entire collection of tunes in Appendix D of Coker (1987). These tunes contain a wide variety of types of modulations and compositional and harmonic devices and demonstrate the general applicability of the harmonic analysis algorithm. The chord changes of the tunes are given by key centers and roman numeral chords. For example, the chord changes for *Solar* (see Figure 1) are represented as in Figure 15. The key centers below the roman numeral chords represent how a

Figure 16. Example of different segmentation due to handling of dominants.

musician might perform tonality segmentation for the tune. They provide a benchmark against which the output of the tonality segmentation algorithm can be compared.

This representation is converted into an ordinary chord chart (such as the one in Figure 1), and then used as input for the structural analysis algorithm and tonality segmentation algorithm. The resulting segmentation and original segmentation are then compared.

Among the 78 tunes in that collection, 57 receive exactly the same segmentation by both Coker and the tonality segmentation algorithm. An additional 13 (tunes 7, 12, 27, 28, 29, 33, 53, 63, 69, 70, 73, 78, and 82) have identical sequences of key centers, but different assignment of dominant chords to consecutive segments. An example of this type of discrepancy is shown in Figure 16, which results from the requirement of the tonality segmentation algorithm to end segments with AEs that do not function as dominant chords. Dominant chords at the boundary of two segments often play dual roles in the two key centers and it is simpler and more consistent to associate them with the chords into which they resolve.

Among the eight remaining tunes, in five (20, 23, 36, 52, and 62) the tonality segmentation algorithm detects more segments than those given by Coker (1987). For example, the tonality segmentation algorithm determines that tune 23 (Charlie Parker's *Blues for Alice*) contains one bar in F major, four bars in B♭ major, two bars in A♭ major, and five bars in F major. Coker considers the entire tune to be in F major. Both analyses are in some sense "correct."

The tonality segmentation algorithm omits some segments in the other three tunes (3, 25, and 54). These segments are too short or fail to satisfy the criteria (they do not contain a root AE of the key center, for example) to be detected by the algorithm.

An OCaml (Leroy 2008) implementation of the structural analysis and tonality segmentation algorithms and test data can be downloaded from the Web page www.sixthhappiness.ca/jazz-harmonic-analysis.

Summary

A new algorithm for harmonic analysis of jazz chord sequences has been described. It views harmonic analysis as a problem of segmenting the input chord sequences and determining the key centers of the segments. This representation is natural and commonly used by jazz musicians. More importantly, it allows modulations in the chord sequences to be modeled explicitly. The harmonic analysis problem can then be formulated mathematically and solved by dynamic programming as an optimization problem. Jazz theory knowledge is incorporated into the algorithm to specify and solve the tonality segmentation problem. Once the segments and key centers are determined, structural information can be recovered from straightforward detection of well-understood elements of jazz theory such as dominant resolutions, harmonic rhythm, substitute dominants, related IIm7s, extended dominants, turnarounds, and interpolated dominants. The algorithm can be used in software that simulates jazz improvisation and for implementation of compositional and teaching tools. An example of the latter is a GUI tool called T2G, which was used to generate the analyses in this article (see Figures 2, 3, 5–7, 9, and 11–14). T2G is also available for download at the URL given at the end of the previous section.

Future work on the tonality segmentation problem can focus on algorithm evaluation and comparison using more comprehensive test data sets, and improved models of modulations (including the use of statistical models).

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